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Q-soliton solution for two-dimensional q-Toda lattice

The Toda lattice is a non-linear evolution equation describing an infinite system of masses on a line that interacts through an exponential force. The paper analyzes the construction of soliton solution for the q-Toda lattice in the two-dimensional case. For this purpose, the equation of motion is taken and the transformation of the dependent variable is used to convert the nonlinear equation into a bilinear form, which is written as the Hirota polynomial. As one of the most effective methods for constructing multisoliton solutions of integrable nonlinear evolution equations, Hirota method is applicable to a wide class of equations, including nonlinear differential, nonlinear differential-difference equations. Using the Hirota method, the bilinear form was obtained for the two-dimensional q-Toda lattice on the basis of which the q-soliton solution was found. The dynamics of the q-soliton solution for two-dimensional q-Toda lattice is presented. Note that the soliton is conserved due to the equilibrium between the action of the nonlinear environment with dispersion. In addition, the soliton behaves like a particle: does not collapse when interacting with each other or other disturbances, while maintaining the structure and continues to move. This quality has the ability to use when transferring data or information over long distances with virtually no interference. In addition, the study of the Toda lattice and the application to it of different methods in different dimensions allows one to proceed to the understanding of such complex terms as matrix models that can be used to describe different physical systems.

Keywords: dispersion, soliton, Toda lattice, bilinear form, Hirota method.

Introduction

Waves described by different nonlinear differential equations, which consist of special pulses, have the property of preserving their original shape like stable particles. They are called solitary waves, single wave particles or solitons. Nonlinear lattices or lattices also contain solitons. When the energy is not very large, nonlinear lattices behave periodically, so stable pulses propagate in such nonlinear continuous systems. The fact of the existence of such lattices shows that there must be some non-linear lattice that allows strict periodic waves, and certain impulses will be stable. One such example is the Toda lattice equation. The Toda lattice is a non-linear evolution equation describing an infinite system of masses on a line that interacts through an exponential force. The Toda lattice is considered as a simple model of the nonlinear one-dimensional crystal in solid state physics. It is defined by a lattice of particles with the interaction of the nearest neighbor, described by the equations of motion [1].

To find the exact solutions for nonlinear differential equations, a huge number of methods are used, such as the Backlund transform [2], the Hirota method [3], the inverse scattering transform method [4], and others. One of the most effective methods for constructing soliton solutions of integrable nonlinear evolution equations is the direct Hirota method, which can be found in [3]. This method is applicable to a wide class of equations, including nonlinear differential, nonlinear differential-difference equations [5-7]. The initial step in this method is to use the transformation of the dependent variable to convert nonlinear partial differential equation into a quadratic form, the so-called bilinear form. The main idea of the method is to write the bilinear form as a Hirota polynomial - D. This compact form is called Hirota's bilinear form. It should be noted that nonlinear partial differential or differential-difference equations can have not only Hirota bilinear forms but also trilinear or multilinear forms [8]. It is assumed that all fully integrable nonlinear partial differential equations or difference equations can be written in Hirota bilinear form. On another hand, for an equation that admits Hirota's bilinear form, the existence of N-soliton solutions of any order is not guaranteed. The equations admitting Hirota's bilinear form and having N-soliton solutions are called integrable by Hirota [9].

In this paper, we present a two-dimensional q-Toda lattice. A one-dimensional case for this equation was studied in [10]. Using the Hirota bilinear method, we find the bilinear form for the two-dimensional q-Toda lattice. Dispersion relation and the q-soliton solution are obtained by bilinear form for the two-dimensional q-Toda lattice.

Two-dimensional q-Toda lattice

In the beginning, classical mechanics was studied specifically for one-dimensional lattices, where the particles forming them interact only with their nearest neighbors. If we restrict their consideration to homogeneous systems, then the mass of each particle is denoted by m, the displacement of the $n-$th particle y_n and the interaction potential between neighboring particles is $\varphi(y_{(n+1)} - y_n)$. Then the equation of motion takes the following form

$$m\frac{d^2 y_n}{dt^2} = \varphi'\left(y_{(n+1)} - y_n\right) - \varphi'\left(y_n - y_{(n-1)}\right)(n = \ldots,-1,0,1,2\ldots), \qquad (1)$$

where φ' derivative φ. Thus,

$$f(r) = -\varphi'(r) = -\frac{d\varphi(r)}{dr}, \qquad (2)$$

$f(r)$ is a force which the spring acts, stretched by the value of r_n

$$r_n = y_{n+1} - y_n \text{ or } \qquad r_n = y_n - y_{n-1} \qquad (3)$$

(3) – this is a relative displacement. When the force $f(r)$ is proportional to the displacement r_n Hooke's law is satisfied. The Toda equation [10], describing the motion of the anharmonic lattice, has the form

$$m\frac{d^2 y_n}{dt^2} = a[e^{br_n} - e^{-br_n}], \qquad (4)$$

where a, b and m are real constants. Introducing the force of the $n-$th particle into the lattice, we obtain the following equation

$$V_n = a[e^{br_n} - e^{-br_n}], \qquad (5)$$

as a rapidly decreasing function, equation (4) turns out to be

$$\frac{d^2}{dt^2} ln(1 + V_n) = V_{(n+1)} + V_{(n-1)} - 2V_n. \qquad (6)$$

The two-dimensional q-Toda lattice has the following form

$$\frac{d^2}{dxdt} ln(1 + V(x,y,t)) = \Delta_x^2 V(x,y,t) = V(x,qy,t) + V\left(x,\frac{y}{q},t\right) - 2V(x,y,t). \qquad (7)$$

Present the transformation of the dependent variable as

$$V(x,y,t) = \frac{d^2}{dxdt}\ln\big(f(x,y,t)\big). \qquad (8)$$

Substituting (8) into (7) and integrating the obtained expression twice, we get

$$\frac{f_{xt} - f_t f_x}{f^2} = \frac{f(x,qy,t) f\left(x,\frac{y}{q},t\right)}{f^2} - 1. \qquad (9)$$

Equation (9) can be rewritten in the Hirota bilinear form, namely in terms of the Hirota D-operator, as

$$\left[D_x D_t - \left(e^{hyD_y} + e^{-hyD_y} - 2\right)\right]\{f(x,y,t)\cdot f(x,y,t)\} = 0, \qquad (10)$$

which follows from (9) by multiplying by $2f^2(x,y,t)$, where we use the q-exponential identity [10]. For functions $f(y), g(y)$ the q-exponential unit [10] will be

$$e^{hyD_y} f(y) g(y) = f(qy) g\left(\frac{y}{q}\right) = E_q f(y) E_q^{-1} g(y),\ y \in R. \qquad (11)$$

The last equation is satisfied if we have the usual relation between two quantum parameters $\hbar$ and q for $q = e^h$. To find the soliton solutions of the Toda lattice, we apply the expansion of perturbations around the formal perturbation parameter ε in the form

$$f(x,y,t) = 1 + \epsilon f^{(1)}(x,y,t) + \varepsilon^2 f^{(2)}(x,y,t) + \cdots. \qquad (12)$$

Substituting (12) into (10) we obtain the equation

$$\begin{gathered} P(D)\{f(x,y,t)\cdot f(x,y,t)\} = P(D)[\{1\cdot 1\} + \varepsilon\{1\cdot f^{(1)} + f^{(1)}\cdot 1\} \\ +\varepsilon^2\{1\cdot f^{(2)} + f^{(2)}\cdot 1 + f^{(1)}\cdot f^{(1)}\} + \varepsilon^3\{1\cdot f^{(3)} + f^{(3)}\cdot 1 + f^{(1)}\cdot f^{(2)} + f^{(2)}\cdot f^{(1)}\} \\ +\varepsilon^4\{1\cdot f^{(4)} + f^{(4)}\cdot 1 + f^{(1)}\cdot f^{(3)} + f^{(3)}\cdot f^{(1)} + f^{(2)}\cdot f^{(2)}\} + \cdots], \end{gathered} \qquad (13)$$

where $P(D) = D_x D_t - \left(e^{hyD_y} + e^{-hyD_y} - 2\right)$. We collect the coefficients with respect to ε^i, $\forall i \geq 0$ of equation (13). The coefficient of the first term ε^0 disappears trivially, and from the coefficient ε^1 we have

$$P(D)\{1 \cdot f^{(1)} + f^{(1)} \cdot 1\} = 2P(\partial) = 2\left[\partial_x \partial_y - \left(e^{hyD_y} + e^{-hyD_y} - 2\right)\right] f^{(1)} = 0. \quad (14)$$

The equation (14) is a direct result of the property of the Hirota operator D [9] because $P(D)$ has an even order. The next important step in the calculation is to find solution for equation (14).

The general trend for soliton solutions is exponential, but the exponential function $f^{(1)}$ does not satisfy equation (14). Due to the nature of the q-numbers, the solution to equation (14) should have a power function for the analog of the q-discrete spatial variable. Therefore, you can choose the original solution (14) as

$$f^{(1)}(x, y, t) = y^{\alpha} e^{\beta t + \gamma x + \eta}, \quad (15)$$

where, α, β, η – arbitrary constants.

A solution with the usual behavior of soliton and having power analogs for q-discrete variables is called a q-soliton solution. If we substitute (15) in (14), we obtain the relation between the parameters

$$\beta\gamma = q^{\alpha} + q^{-\alpha} - 2, \quad (16)$$

which is called the dispersion relation.

The coefficient ε^2 obtained from (13) gives the following

$$P(D)\{1 \cdot f^{(2)} + f^{(2)} \cdot 1 + f^{(1)} \cdot f^{(1)}\} = 2P(\partial) f^{(2)} + P(D)\{f^{(1)} \cdot f^{(1)}\}.$$

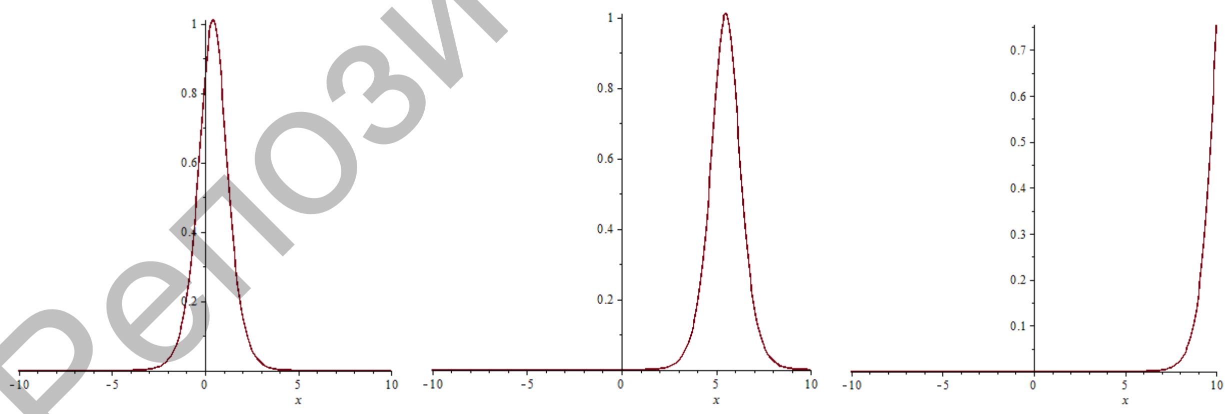
That gives

$$\left[D_x D_t - \left(e^{hyD_y} + e^{-hyD_y} - 2\right)\right]\{f^{(1)}(x, y, t) \cdot f^{(1)}(x, y, t)\}$$
$$= -2\left[\partial_x \partial_t - \left(e^{hy\partial_y} + e^{-hy\partial_y} - 2\right)\right] f^{(2)}(x, y, t). \quad (17)$$

Since $f^{(1)}$ given in (14) satisfies to form (17), we can assume that all members of higher order are zero, i.e. $f^{(j)} = 0, j \geq 2$. Further, as a generalization, this fact can be assumed in the derivation of the i-q-soliton solution, $f^{(j)} = 0$ for all $j \geq i + 1$. When $\varepsilon = 1$, one-q-soliton solution is constructed by substituting equations (15) and (16) into (17) and taking into account that $f(x, y, t) = 1 + f^{(1)}(x, y, t)$ then

$$V(x, y, t) = \frac{d^2}{dxdt} \ln f(x, y, t) = \frac{y^{\alpha} \beta\gamma e^{\beta t + \gamma x + \eta}}{\left(1 + y^{\alpha} e^{\beta t + \gamma x + \eta}\right)^2}, \quad (18)$$

which is the one-q-soliton solution of the two-dimensional q-Toda lattice. The dynamics of the one-q-soliton solution is presented in Figure.



a) b) c)

Figure. Dynamics of a one-q-soliton solution for the two-dimensional q-Toda lattice with parameters: γ=2, α=-5, η=0, q=1.25, (a) t=-5, (b) t=0, (c) t=5.

Figure shows the dynamics of the obtained solution (18) depending on t. So with different values of t, the wave's shape is saved. This proves there is a soliton in a two-dimensional q-Toda lattice, which means that energy transfer is possible. The Toda lattice is unique because it has a wide range from the harmonic to the anharmonic limit and has the so-called N-soliton solutions. As presented above, soliton is a structurally unchanged solitary wave in a nonlinear environment. When interacting with each other or other disturbances, solitons behave like particles, therefore they are called particle-like. Due to the balance between the action of nonlinearity and dispersion, they save their structure, not collapsing in a collision.

Conclusion

Thus, we present the q-Toda lattice in the two-dimensional case. Using the Hirota bilinear method, we find the bilinear form for the two-dimensional q-Toda lattice and obtain the dispersion relation and the one-q-soliton solution. This algorithm can be applied to obtain N-soliton solutions. The soliton is conserved due to the equilibrium between the action of a nonlinear environment with dispersion. In addition, the soliton behaves as a particle (particle-like): it does not collapse when interacting with each other or other disturbances while maintaining the structure and continues to move. This quality has the ability to use when transferring data or information over long distances with virtually no interference. The study of the Toda lattice in various dimensions allows one to go on to understand such complex terms as matrix models that can be used to describe different physical systems.

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Екіөлшемді q-Тода тізбегінің q-солитондық шешімі

Тода тізбегі экспоненциалдық күш арқылы өзара әрекеттесетін, сызықтағы шексіз массалар жүйесін сипаттайтын сызықты емес эволюциялық теңдеу болып табылады. Авторлар q-Тода тізбегінің екіөлшемді кеңістіктегі солитондық шешімін құрастыруды талдады. Мақсатқа қолжеткізу үшін қозғалыс теңдеуі алынып, Хирота полиномы ретінде жазылған бисызықты түрге келтіру үшін сызықты емес теңдеуді тәуелді айнымалыны түрлендіру қолданды. Интегралданатын сызықты емес эволюциялық теңдеулердің көп солитонды шешімдерін құрастырудың тиімді әдістерінің бірі ретінде берілген әдісті көптеген теңдеулерге, олардың ішінде сызықты емес дифференциал, сызықты емес дифференциал-айырымдық теңдеулерге қолдануға болады. Хирота әдісін қолдана отырып, екіөлшемді q-Тода тізбегінің бисызықты түрі алынып, оның негізінде q-солитондық шешімі есептелді. Екіөлшемді q-Тода тізбегінің q-солитондық шешімнің динамикасы ұсынылды. Атап айтқанда, солитон сызықты емес орта мен дисперсия арасындағы әрекетінің тепе-теңдігі арқасында сақталады. Сонымен қатар солитон өзін бөлшек ретінде ұстайды: бір-бірімен немесе басқа ауытқулармен өзара әрекеттесу кезінде қирамай, құрылымын сақтап, қозғалысын жалғастырады. Осындай қасиетті мәліметті немесе ақпаратты алысқа дерлік кедергісіз жіберу кезінде қолдануға мүмкіндік туғызады. Бұдан басқа, Тода тізбегін және оған түрлі өлшемдіктегі әртүрлі әдістердің қолдануын зерттеу әркелкі физикалық жүйелерді сипаттау мақсатында қолдануға болатын, матрицалық модельдер сияқты күрделі терминдерді түсінуге мүмкіндік туғызады.

Кілт сөздер: дисперсия, солитон, Тода тізбегі, бисызықты форма, Хирота әдісі.

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Q-солитонное решение двумерной цепочки q-Тоды

Цепочка Тоды является нелинейным эволюционным уравнением, описывающим бесконечную систему масс на линии, которые взаимодействуют через экспоненциальную силу. В работе проведен анализ построения солитонного решения цепочки q-Тоды в двумерном пространстве. Для этой цели взято уравнение движения и использовано преобразование зависимой переменной для преобразования нелинейного уравнения в билинейную форму, которая записана как полином оператора Хироты. Как один из наиболее эффективных методов построения многосолитонных решений интегрируемых нелинейных эволюционных уравнений, данный метод применим к широкому классу уравнений, включая нелинейные дифференциальные, нелинейные дифференциально-разностные уравнения. Применяя метод Хироты, была получена билинейная форма для двумерной цепочки q-Тоды на основе, которой найдено q-солитонное решение. Представлена динамика q-солитонного решения двумерной цепочки q-Тоды. Отметим, что солитон сохраняется благодаря равновесию между действием нелинейной среды с дисперсией. Помимо этого солитон ведет себя как частица: не разрушается при взаимодействии друг с другом или другими возмущениями, при этом сохраняет структуру и продолжает движение. Такое качество имеет возможность использования при передаче данных или информации на дальние расстояния практически без помех. Кроме того, исследование цепочки Тоды и применение к ней разных методов в различных размерностях позволяет перейти к пониманию таких сложных терминов, как матричные модели, которые можно применить для описания разных физических систем.

Ключевые слова: дисперсия, солитон, цепочка Тоды, билинейная форма, метод Хироты.

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