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Estimates of the norm of the convolution operator in anisotropic Besov spaces with the dominated mixed derivative

In this paper, we investigate the boundedness of the norm of the convolution operator in Sobolev spaces with the dominated mixed derivative and anisotropic Nikolsky-Besov spaces. For Sobolev spaces with the dominated mixed derivatives, an analogue of Young's inequality is obtained, namely, relations of the form

$$W_{\mathbf{p}}^{\gamma} * W_{\mathbf{r}}^{\beta} \hookrightarrow W_{\mathbf{q}}^{\alpha} \quad (1)$$

are proved when the corresponding conditions on the parameters are satisfied. The main goal of the paper is to solve the following problems. Let f and g be functions from some classes of the Nikolsky-Besov space scale. We would like to find the Nikolsky-Besov space such that the convolution $f * g$ belongs to this space. Using relation (1) and the Nursultanov interpolation theorem for anisotropic spaces, an analogue of the O'Neil theorem was obtained for the Nikolsky-Besov space scale $B_{\mathbf{p}\mathbf{q}}^{\alpha}$, where α , \mathbf{p} , \mathbf{q} are vector parameters. Relations of the form $B_{\mathbf{p}\mathbf{s}_1}^{\gamma} * B_{\mathbf{r}\mathbf{s}_2}^{\beta} \hookrightarrow B_{\mathbf{q}\mathbf{s}}^{\alpha}$ are obtained, with the corresponding ratios of vector parameters. The theorems obtained in this paper complement the results of Batyrov and Burenkov, where similar problems were considered in isotropic Nikolsky-Besov spaces, that is, in spaces where the parameters are scalars.

Keywords: convolution operator, anisotropic Sobolev and Besov spaces, interpolation.

Introduction

Let I be either a n -dimensional torus $\mathbb{T}^n = [0, 1]^n$, or the Euclidean space \mathbb{R}^n . Let $f(x)$ and $g(x)$ be measurable functions on I with respect to the n -dimensional Lebesgue measure such that for almost all $x \in I$ there exists an integral

$$\int_I f(x-y)g(y)dy.$$

In this case, it is said that the convolution of these functions is defined

$$(f * g)(x) = \int_I f(x-y)g(y)dy. \quad (1.1)$$

The classical Young's inequality [1; 199] has the following form. Let

$$1 \leq p, r, q \leq \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}. \quad (1.2)$$

If $f \in L_p(I)$, $g \in L_r(I)$, then there exists almost everywhere on I the convolution $f * g$, belonging to the space $L_q(I)$ and the following inequality holds

$$\|f * g\|_{L_q(I)} \leq \|f\|_{L_p(I)} \|g\|_{L_r(I)}. \quad (1.3)$$

We will write this statement as follows

$$L_p(I) * L_r(I) \hookrightarrow L_q(I).$$

This inequality plays an important role in harmonic analysis and in the theory of partial differential equations [1–3].

Note that if

$$1 < p, r, q < \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}, \quad (1.4)$$

then for $g_0(x) = \frac{1}{|x|^{\frac{n}{r}}}$ the inequality holds

$$\|f * g_0\|_{L_q(I)} \leq C \|f\|_{L_p(I)}.$$

This inequality is called the Hardy-Littlewood-Sobolev inequality. It does not follow from Young's inequality, since $\|g_0\|_{L_r(I)} = \infty$. A generalization of inequality (1.3) obtained by O'Neil [4] (see also [5, 6]).

If (1.4) is true and $0 < s_1, s_2, s \leq \infty, \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$, then

$$L_{ps_1} * L_{rs_2} \hookrightarrow L_{qs} \quad (1.5)$$

and in particular

$$L_p * L_{r\infty} \hookrightarrow L_q, \quad (1.6)$$

where L_{ps} is Lorentz space.

Note that in relation (1.5), condition (1.4) is essential. The limiting cases of the O'Neil inequality with condition (1.2) were considered in [7].

The O'Neil inequality for anisotropic Lorentz spaces was studied in [8–10]. In the case of $n \geq 2$ these results are extend the inequality (1.6). In the one-dimensional case, the O'Neil inequality was extended in [11, 12].

There are generalizations of the Young and O'Neil inequalities for various functional spaces: weighted L_p spaces, classical and Lorentz weighted spaces, Hardy spaces, Wiener spaces, Orlicz spaces; [5, 6, 8, 13–18], and references therein.

Convolution operators were studied in various spaces of smooth functions in [19–22].

V.I. Burenkov and B.E. Batyrov in [21] proved the following statement: let $-\infty < l_1, l_2, l_3 < \infty, 0 < p_1, p_2, p_3 \leq \infty, 0 < \theta_1, \theta_2, \theta_3 \leq \infty$. For any $f_1 \in B_{p_1\theta_1}^{l_1}(\mathbb{R}^n), f_2 \in B_{p_2\theta_2}^{l_2}(\mathbb{R}^n)$ such that Ff_1 and Ff_2 are regular generalized functions and their (pointwise) product $Ff_1 \cdot Ff_2 \in S'(\mathbb{R}^n)$, there exists a number $c_3 > 0$ such that

$$\|f_1 * f_2\|_{B_{p_3\theta_3}^{l_3}(\mathbb{R}^n)} \leq c_3 \|f_1\|_{B_{p_1\theta_1}^{l_1}(\mathbb{R}^n)} \|f_2\|_{B_{p_2\theta_2}^{l_2}(\mathbb{R}^n)}, \quad (1.7)$$

holds if and only if the following conditions hold:

- 1) $p_3 \geq p_1, p_3 \geq p_2$;
- 2) $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 \geq 0$;

and one of the conditions

$$3a) \quad l_3 < l_1 + l_2 - n \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 \right)$$

or

$$3b) \quad l_3 = l_1 + l_2 - n \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 \right) \text{ и } \frac{1}{\theta_3} \leq \frac{1}{\theta_1} + \frac{1}{\theta_2},$$

where Ff is the Fourier transform of the function f :

$$(Ff)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(\xi) d\xi.$$

For $p_2 = p_3, \theta_2 = \theta_3, 0 < l_2 < l_3 < \infty$ inequality (1.7) and some of its generalizations follow from the results obtained in the works of K.K. Golovkin and V.A. Solonnikov [19, 20], and [23].

In this paper, we investigate the boundedness of the convolution operator in anisotropic Besov spaces with the dominated mixed derivative.

2. Anisotropic Besov spaces with dominated mixed derivative

Let $\alpha \in \mathbb{R}^n$, $\mathbf{1} < \mathbf{p} = (p_1, \dots, p_n) < \infty$, $\mathbf{0} < \mathbf{q} = (q_1, \dots, q_n) \leq \infty$. Following [24–26], we define the space $B_{\mathbf{p}\mathbf{q}}^\alpha(\mathbb{T}^n)$ as the set of series $f = \sum_{m \in \mathbb{Z}^n} a_m e^{2\pi i(m,x)}$ (generally speaking, divergent) for which

$$\|f\|_{B_{\mathbf{p}\mathbf{q}}^\alpha(\mathbb{T}^n)} = \left(\sum_{k_n=0}^{\infty} \dots \left(\sum_{k_1=0}^{\infty} \left(2^{\sum_{j=1}^n \alpha_j k_j} \|\Delta_k(f)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}} < \infty$$

is finite, where $\Delta_k(f)(x) = \sum_{\substack{2^{k_j-1} \leq |m_j| < 2^{k_j} \\ j=1, \dots, n}} a_m e^{2\pi i(m,x)}$, $k \in \mathbb{Z}_+^n$, $(m, x) = \sum_{i=1}^n m_i x_i$.

For $q = \infty$ the values of $\left(\sum_{k \in \mathbb{Z}} b_k^q \right)^{\frac{1}{q}}$, $\left(\int_{\mathbb{T}} f^q \right)^{\frac{1}{q}}$ are understood respectively as $\sup_{k \in \mathbb{Z}} |b_k|$, $\text{ess sup}_{x \in \mathbb{T}} |f(x)|$.

We say that the series $f = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i(k,x)}$ is an element of the space $W_{\mathbf{p}}^\alpha(\mathbb{T}^n)$ [24] if there is a

function $f^\alpha \in L_{\mathbf{p}}(\mathbb{T}^n)$ Fourier series of which coincides with the series $\sum_{k \in \mathbb{Z}^n} \bar{k}^\alpha a_k e^{2\pi i(k,x)}$, here $\bar{k}^\alpha = \prod_{j=1}^n \bar{k}_j^\alpha$, $\bar{k}_j = \max\{|k_j|, 1\}$, $j = 1, \dots, n$,

$$\|f\|_{W_{\mathbf{p}}^\alpha(\mathbb{T}^n)} \stackrel{\text{def}}{=} \|f^\alpha\|_{L_{\mathbf{p}}(\mathbb{T}^n)}.$$

We define the concept of convolution for the elements of these spaces.

Let $f = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i(k,x)}$ and $g = \sum_{k \in \mathbb{Z}^n} b_k e^{2\pi i(k,x)}$ be trigonometric series. By the convolution of these series we mean the series

$$f * g = \sum_{k \in \mathbb{Z}^n} a_k b_k e^{2\pi i(k,x)}. \tag{2.1}$$

Note that for the «good» functions f and g , the convolution defined by equality (2.1) coincides with the classical definition (1.1). If the functions f and g from the corresponding spaces in (1.3), then $f(x) \stackrel{L_{\mathbf{p}}}{=} \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i(k,x)}$ and $g(x) \stackrel{L_{\mathbf{r}}}{=} \sum_{k \in \mathbb{Z}^n} \hat{g}(k) e^{2\pi i(k,x)}$ and $(f * g)(x) = \int_{\mathbb{T}^n} f(x-y)g(y)dy = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\hat{g}(k) e^{2\pi i(k,x)}$. Here, equalities are understood in the sense of the corresponding metrics.

We will need interpolation properties of anisotropic Sobolev and Besov spaces [24, 27]. Let $0 < \theta = (\theta_1, \dots, \theta_n) < 1$, $E = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) : \varepsilon_j \in \{0, 1\}, j = 1, \dots, n\}$ be the vertices of the n -dimensional unit cube, $\{A_\varepsilon\}_{\varepsilon \in E}$ be Banach spaces that are subspaces of some linear Hausdorff space. For the element $a \in \sum_{\varepsilon \in E} A_\varepsilon$, we define the functional

$$K(t, a; A_\varepsilon, \varepsilon \in E) = \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} t^\varepsilon \|a_\varepsilon\|_{A_\varepsilon},$$

where $t^\varepsilon = t_1^{\varepsilon_1} \dots t_n^{\varepsilon_n}$.

By $A_{\theta\mathbf{q}} = (A_\varepsilon; \varepsilon \in E)_{\theta\mathbf{q}}$ we denote a linear subset of $\sum_{\varepsilon \in E} A_\varepsilon$, for elements of which

$$\|a\|_{A_{\theta\mathbf{q}}} = \left(\int_0^\infty \dots \left(\int_0^\infty |t_1^{\frac{1}{\theta_1}-1} \dots t_n^{\frac{1}{\theta_n}-1} K(t, a; A_\varepsilon, \varepsilon \in E)|^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n} \right)^{\frac{1}{q_n}} < \infty$$

is true.

Lemma 2.1 [28] Let T be a linear operator such that

$$T : A_\varepsilon \rightarrow B_\varepsilon \text{ with norm } M_\varepsilon, \varepsilon \in E.$$

Then

$$T : (A_\varepsilon; \varepsilon \in E)_{\theta \mathbf{q}} \rightarrow (B_\varepsilon, \varepsilon \in E)_{\theta \mathbf{q}}$$

with the norm $\|T\| \leq \max_{\varepsilon \in E} M_\varepsilon$.

Theorem 2.1 ([24]) Let $\mathbf{1} \leq \mathbf{p} = (p_1, \dots, p_n) < \infty$, $\mathbf{0} < \mathbf{r} = (r_1, \dots, r_n)$, $\mathbf{q} = (q_1, \dots, q_n) \leq \infty$, $\varepsilon \in E$, $\alpha_0 = (\alpha_1^0, \dots, \alpha_n^0)$, $\alpha_1 = (\alpha_1^1, \dots, \alpha_n^1) \in \mathbb{R}^n$. Then

$$(B_{\mathbf{pr}}^{\alpha_\varepsilon}(\mathbb{T}^n); \varepsilon \in E)_{\theta \mathbf{q}} = B_{\mathbf{pq}}^\alpha(\mathbb{T}^n),$$

$$(W_{\mathbf{p}}^{\alpha_\varepsilon}(\mathbb{T}^n); \varepsilon \in E)_{\theta \mathbf{q}} = B_{\mathbf{pq}}^\alpha(\mathbb{T}^n),$$

where $\alpha_\varepsilon = (\alpha_1^\varepsilon, \dots, \alpha_n^\varepsilon)$, $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) < \mathbf{1}$, $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$.

3. Main result

Lemma 3.1 Let $\mathbf{1} \leq \mathbf{q}, \mathbf{p}, \mathbf{r} < \infty$, $\frac{1}{\mathbf{q}} + \mathbf{1} = \frac{1}{\mathbf{p}} + \frac{1}{\mathbf{r}}$, $\alpha, \beta, \gamma \in \mathbb{R}^n$, $\alpha = \beta + \gamma$. Suppose that $f \in W_{\mathbf{p}}^\beta(\mathbb{T}^n)$, $g \in W_{\mathbf{r}}^\gamma(\mathbb{T}^n)$. Then $f * g \in W_{\mathbf{q}}^\alpha(\mathbb{T}^n)$ and

$$\|f * g\|_{W_{\mathbf{q}}^\alpha(\mathbb{T}^n)} \leq \|f\|_{W_{\mathbf{p}}^\beta(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^\gamma(\mathbb{T}^n)}.$$

Proof. Let $f = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i(k,x)} \in W_{\mathbf{p}}^\beta(\mathbb{T}^n)$, $g = \sum_{k \in \mathbb{Z}^n} b_k e^{2\pi i(k,x)} \in W_{\mathbf{r}}^\gamma(\mathbb{T}^n)$. According to the definition, there are functions $f^\beta \in L_{\mathbf{p}}(\mathbb{T}^n)$, $g^\gamma \in L_{\mathbf{r}}(\mathbb{T}^n)$ Fourier series of which coincide, respectively, with the $\sum_{k \in \mathbb{Z}^n} \bar{k}^\beta a_k e^{2\pi i(k,x)}$, $\sum_{k \in \mathbb{Z}^n} \bar{k}^\gamma b_k e^{2\pi i(k,x)}$.

From Young's inequality for Lebesgue spaces with mixed metric [29; 25] $(f^\beta * g^\gamma) \in L_{\mathbf{q}}(\mathbb{T}^n)$ and has the inequality

$$\|f^\beta * g^\gamma\|_{L_{\mathbf{q}}(\mathbb{T}^n)} \leq \|f^\beta\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \|g^\gamma\|_{L_{\mathbf{r}}(\mathbb{T}^n)}.$$

Now we note that

$$(f^\beta * g^\gamma)(x) \stackrel{L_{\mathbf{q}}}{=} \sum_{k \in \mathbb{Z}^n} \bar{k}^{\beta+\gamma} a_k b_k e^{2\pi i(k,x)} = (f * g)^\alpha(x),$$

which means that $(f * g) \in W_{\mathbf{q}}^\alpha(\mathbb{T}^n)$ and the inequality

$$\|f * g\|_{W_{\mathbf{q}}^\alpha(\mathbb{T}^n)} \leq \|f\|_{W_{\mathbf{p}}^\beta(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^\gamma(\mathbb{T}^n)}$$

holds.

Theorem 3.1 Let $\alpha, \beta, \gamma \in \mathbb{R}^n$, $\alpha = \beta + \gamma$, $\mathbf{1} \leq \mathbf{q}, \mathbf{p}, \mathbf{r} < \infty$, $\mathbf{1} + \frac{1}{\mathbf{q}} = \frac{1}{\mathbf{p}} + \frac{1}{\mathbf{r}}$, $\mathbf{0} < \mathbf{h}, \eta, \xi \leq \infty$, $\frac{1}{\mathbf{h}} = \frac{1}{\eta} + \frac{1}{\xi}$. Suppose that $f \in B_{\mathbf{p}\eta}^\beta(\mathbb{T}^n)$, $g \in B_{\mathbf{r}\xi}^\gamma(\mathbb{T}^n)$. Then $f * g \in B_{\mathbf{q}\mathbf{h}}^\alpha(\mathbb{T}^n)$ and

$$\|f * g\|_{B_{\mathbf{q}\mathbf{h}}^\alpha(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\eta}^\beta(\mathbb{T}^n)} \|g\|_{B_{\mathbf{r}\xi}^\gamma(\mathbb{T}^n)}.$$

Proof. Let $f \in W_{\mathbf{p}}^\beta(\mathbb{T}^n)$, $g \in W_{\mathbf{r}}^\gamma(\mathbb{T}^n)$, then from Lemma 3.1 it follows that $(f * g) \in W_{\mathbf{q}}^\alpha(\mathbb{T}^n)$ and the following inequality

$$\|f * g\|_{W_{\mathbf{q}}^\alpha(\mathbb{T}^n)} \leq \|f\|_{W_{\mathbf{p}}^\beta(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^\gamma(\mathbb{T}^n)} \quad (3.1)$$

holds true.

Let $\alpha_0 = (\alpha_1^0, \dots, \alpha_n^0)$, $\alpha_1 = (\alpha_1^1, \dots, \alpha_n^1)$, $\beta_0 = (\beta_1^0, \dots, \beta_n^0)$, $\beta_1 = (\beta_1^1, \dots, \beta_n^1) \in \mathbb{R}^n$, $\alpha_i^0 \neq \alpha_i^1$, $\beta_i^0 \neq \beta_i^1$, $i = \overline{1, n}$. Let $\alpha_\varepsilon = (\alpha_1^\varepsilon, \dots, \alpha_n^\varepsilon)$, $\beta_\varepsilon = (\beta_1^\varepsilon, \dots, \beta_n^\varepsilon)$, $\varepsilon \in E$ such that $\alpha_\varepsilon - \beta_\varepsilon = \gamma$.

We rewrite inequality (3.1) for the α_ε and β_ε parameters

$$\|f * g\|_{W_{\mathbf{q}}^{\alpha_\varepsilon}(\mathbb{T}^n)} \leq \|f\|_{W_{\mathbf{p}}^{\beta_\varepsilon}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^\gamma(\mathbb{T}^n)}, \quad \alpha_\varepsilon = \beta_\varepsilon + \gamma, \quad \varepsilon \in E.$$

For a fixed $g \in W_{\mathbf{r}}^{\gamma}(\mathbb{T}^n)$ operator $A_g f = f * g$ acts boundedly from $W_{\mathbf{p}}^{\beta_{\varepsilon}}(\mathbb{T}^n)$ to $W_{\mathbf{q}}^{\alpha_{\varepsilon}}(\mathbb{T}^n)$. Then, using the anisotropic interpolation theorem (Theorem 2.1)

$$A_g : \left(W_{\mathbf{p}}^{\beta_{\varepsilon}}(\mathbb{T}^n), \varepsilon \in E \right)_{\theta \xi} \rightarrow \left(W_{\mathbf{q}}^{\alpha_{\varepsilon}}(\mathbb{T}^n), \varepsilon \in E \right)_{\theta \xi},$$

we obtain that the operator acts boundedly

$$A_g : B_{\mathbf{p}\xi}^{\beta}(\mathbb{T}^n) \rightarrow B_{\mathbf{q}\xi}^{\alpha}(\mathbb{T}^n)$$

and

$$\|A_g\| \leq C \|g\|_{W_{\mathbf{r}}^{\gamma}(\mathbb{T}^n)},$$

where $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$, $\beta = (1 - \theta)\beta_0 + \theta\beta_1$ for any $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) < \mathbf{1}$. Thus, we have obtained the inequality:

$$\|f * g\|_{B_{\mathbf{q}\xi}^{\alpha}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\xi}^{\beta}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\gamma}(\mathbb{T}^n)}, \quad (3.2)$$

where $\alpha, \beta, \gamma, \mathbf{p}, \mathbf{q}, \mathbf{r}$ satisfy the conditions of the theorem.

In inequality (3.2) we assume that $\xi = \infty$, then we have

$$\|f * g\|_{B_{\mathbf{q}\infty}^{\alpha}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\infty}^{\beta}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\gamma}(\mathbb{T}^n)}.$$

We fix $\beta \in \mathbb{R}^n$. Let $\alpha_0, \alpha_1, \gamma_0, \gamma_1 \in \mathbb{R}^n$ be arbitrary vector parameters satisfying the conditions $\alpha_i = \beta + \gamma_i$, $i = 0, 1$ и $\alpha_j^0 \neq \alpha_j^1, \gamma_j^0 \neq \gamma_j^1, j = \overline{1, n}$. Then for the parameters $\alpha_{\varepsilon} = (\alpha_1^{\varepsilon_1}, \dots, \alpha_n^{\varepsilon_n})$, $\gamma_{\varepsilon} = (\gamma_1^{\varepsilon_1}, \dots, \gamma_n^{\varepsilon_n})$, $\varepsilon \in E$ the inequality

$$\|f * g\|_{B_{\mathbf{q}\infty}^{\alpha_{\varepsilon}}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\infty}^{\beta}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\gamma_{\varepsilon}}(\mathbb{T}^n)}.$$

holds.

Now for a fixed $f(x)$ we define a linear operator $B_f g = f * g$. Then B_f acts boundedly from $W_{\mathbf{r}}^{\gamma_{\varepsilon}}(\mathbb{T}^n)$ in $B_{\mathbf{q}\infty}^{\alpha_{\varepsilon}}(\mathbb{T}^n)$ with an estimate of the norm

$$\|B_f\|_{W_{\mathbf{r}}^{\gamma_{\varepsilon}}(\mathbb{T}^n) \rightarrow B_{\mathbf{q}\infty}^{\alpha_{\varepsilon}}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\infty}^{\beta}(\mathbb{T}^n)}.$$

Further, using Theorem 2.1 and Lemma 2.1, we have that the operator B_f is bounded from $B_{\mathbf{r}\xi}^{\gamma}(\mathbb{T}^n)$ to $B_{\mathbf{q}\xi}^{\alpha}(\mathbb{T}^n)$ and

$$\|f * g\|_{B_{\mathbf{q}\xi}^{\alpha}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\infty}^{\beta}(\mathbb{T}^n)} \|g\|_{B_{\mathbf{r}\xi}^{\gamma}(\mathbb{T}^n)},$$

where $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$, $\gamma = (1 - \theta)\gamma_0 + \theta\gamma_1$ for any $\mathbf{0} < \theta = (\theta_0, \dots, \theta_n) < \mathbf{1}$.

Similarly, we can obtain the inequality

$$\|f * g\|_{B_{\mathbf{q}\eta}^{\alpha}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\eta}^{\beta}(\mathbb{T}^n)} \|g\|_{B_{\mathbf{r}\infty}^{\gamma}(\mathbb{T}^n)}.$$

Thus, for the bilinear convolution operator $T(f, g) = f * g$, we have

$$T : B_{\mathbf{p}\infty}^{\beta_0}(\mathbb{T}^n) \times B_{\mathbf{r}\xi}^{\gamma_0}(\mathbb{T}^n) \rightarrow B_{\mathbf{q}\xi}^{\alpha_0}(\mathbb{T}^n),$$

$$T : B_{\mathbf{p}\eta}^{\beta_1}(\mathbb{T}^n) \times B_{\mathbf{r}\infty}^{\gamma_1}(\mathbb{T}^n) \rightarrow B_{\mathbf{q}\eta}^{\alpha_1}(\mathbb{T}^n),$$

where the corresponding parameters satisfy the conditions of the theorem.

Next, applying the bilinear interpolation theorem (Theorem 4.4.1, [30, 125]), we have

$$T : \left(B_{\mathbf{p}\infty}^{\beta_0}(\mathbb{T}^n), B_{\mathbf{p}\eta}^{\beta_1}(\mathbb{T}^n) \right)_{[\theta]} \times \left(B_{\mathbf{r}\xi}^{\gamma_0}(\mathbb{T}^n), B_{\mathbf{r}\infty}^{\gamma_1}(\mathbb{T}^n) \right)_{[\theta]} \rightarrow \left(B_{\mathbf{q}\xi}^{\alpha_0}(\mathbb{T}^n), B_{\mathbf{q}\eta}^{\alpha_1}(\mathbb{T}^n) \right)_{[\theta]}.$$

Since the space $B_{\mathbf{p}\mathbf{q}}^{\mathbf{s}}(\mathbb{T}^n)$ is a retract of $l_{\mathbf{q}}^{\mathbf{s}}(L_{\mathbf{p}})(\mathbb{T}^n)$, we have

$$\left(B_{\mathbf{p}\mathbf{s}_0}^{\beta_0}(\mathbb{T}^n), B_{\mathbf{p}\mathbf{s}_1}^{\beta_1}(\mathbb{T}^n) \right)_{[\theta]} = B_{\mathbf{p}\mathbf{s}}^{\beta}(\mathbb{T}^n),$$

where $\beta = \beta_0(1 - \theta) + \beta_1\theta$, $\frac{1}{\mathbf{s}} = \frac{1 - \theta}{\mathbf{s}_0} + \frac{\theta}{\mathbf{s}_1}$.

This implies

$$T : B_{\mathbf{p}\eta}^{\beta}(\mathbb{T}^n) \times B_{\mathbf{r}\xi}^{\gamma}(\mathbb{T}^n) \rightarrow B_{\mathbf{q}\mathbf{h}}^{\alpha}(\mathbb{T}^n).$$

Finally,

$$\|f * g\|_{B_{\mathbf{q}\mathbf{h}}^{\alpha}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\eta}^{\beta}(\mathbb{T}^n)} \|g\|_{B_{\mathbf{r}\xi}^{\gamma}(\mathbb{T}^n)},$$

where

$$\frac{1}{\mathbf{h}} = \frac{1}{\eta} + \frac{1}{\xi}, \quad \mathbf{1} + \frac{1}{\mathbf{q}} = \frac{1}{\mathbf{p}} + \frac{1}{\mathbf{r}}, \quad \alpha = \beta + \gamma.$$

Taking into account the embeddings of spaces ([24], Theorem 4), we can obtain the following theorem.

Theorem 3.2 Let $\alpha, \beta, \gamma \in \mathbb{R}^n$, $\alpha \leq \beta + \gamma$, $1 \leq \mathbf{q}, \mathbf{p}, \mathbf{r} < \infty$, $\alpha = \beta + \gamma + \mathbf{1} + \frac{1}{\mathbf{q}} - \frac{1}{\mathbf{p}} - \frac{1}{\mathbf{r}}$, $\mathbf{0} < \mathbf{h}, \eta, \xi \leq \infty$.

Suppose that $f(x)$ and $g(x)$ are measurable functions on \mathbb{T}^n such that $f \in B_{\mathbf{p}\eta}^{\beta}(\mathbb{T}^n)$, $g \in B_{\mathbf{r}\xi}^{\gamma}(\mathbb{T}^n)$. Then $f * g \in B_{\mathbf{q}\mathbf{h}}^{\alpha}(\mathbb{T}^n)$ and

$$\|f * g\|_{B_{\mathbf{q}\mathbf{h}}^{\alpha}(\mathbb{T}^n)} \leq C \|f\|_{B_{\mathbf{p}\eta}^{\beta}(\mathbb{T}^n)} \|g\|_{B_{\mathbf{r}\xi}^{\gamma}(\mathbb{T}^n)},$$

where $\frac{1}{\mathbf{h}} \leq \frac{1}{\eta} + \frac{1}{\xi}$.

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Аралас туындысы басым анизотропты Бесов кеңістігіндегі үйірткі операторының нормаларын бағалау

Мақалада үйірткі операторы нормасының аралас туындысы басым Соболев және анизотропты Никольский-Бесов кеңістіктеріндегі шенелуі зерттелді. Аралас туындысы басым Соболев кеңістігі үшін Юнг теңсіздігінің аналогы алынды, атап айтқанда, $W_p^\gamma * W_r^\beta \hookrightarrow W_q^\alpha$ түріндігі қатынас дәлелденді, мұнда қатынас параметрлеріне сәйкесінше шарттар қойылған. Жұмыстың негізгі мақсаты келесі есеп болып табылады: айталық, f және g — Никольский-Бесов кеңістігі шкаласының қандай да бір функциялар класы болсын. Олардың $f * g$ үйірткісі қай кеңістікке жататынын анықтау қажет. (1) қатынас пен Нурсултановтың анизотропты кеңістіктерге арналған интерполяциялық теоремаларын қолдана отырып, B_{pq}^α Никольский-Бесов кеңістігінің шкаласы үшін О'Нейл теоремасының аналогы алынды, мұнда α, p, q — векторлық параметрлер. $B_{ps_1}^\gamma * B_{rs_2}^\beta \hookrightarrow B_{qs}^\alpha$ түріндігі қатынас дәлелденді, мұнда қатынас параметрлеріне сәйкесінше шарттар қойылған. Осы жұмыста алынған теоремалар изотропты Никольский-Бесов кеңістігіндегі, яғни скаляр параметрлі кеңістіктегі, Батыров пен Буренковтың ұқсас есептер қарастырған нәтижелерін толықтырады.

Кілт сөздер: үйірткі операторы, анизотропты Бесов кеңістігі, анизотропты Соболев кеңістігі, интерполяция.

Оценки нормы оператора свертки в анизотропных пространствах Бесова с доминирующей смешанной производной

В статье исследована ограниченность нормы оператора свертки в пространствах Соболева, с доминирующей смешанной производной, и анизотропных пространствах Никольского-Бесова. Для пространств Соболева с доминирующей смешанной производной получен аналог неравенства Юнга, а именно доказаны соотношения вида $W_p^\gamma * W_r^\beta \hookrightarrow W_q^\alpha$ при выполнении соответствующих условий на параметры. Основной целью работы является решение следующей задачи: пусть f и g — функции из некоторых классов шкалы пространств Никольского-Бесова. Нужно определить, к какому пространству принадлежит их свертка $f * g$. Используя соотношение (1) и интерполяционные теоремы Нурсултанова для анизотропных пространств, получен аналог теоремы О'Нейла для шкалы пространств Никольского-Бесова B_{pq}^α , где α, p, q — векторные параметры. Получены соотношения вида $B_{ps_1}^\gamma * B_{rs_2}^\beta \hookrightarrow B_{qs}^\alpha$, при соответствующих соотношениях векторных параметров. Полученные в данной работе теоремы дополняют результаты Батырова и Буренкова, где рассматривались подобные задачи в изотропных пространствах Никольского-Бесова, т.е. в пространствах, где параметры являются скалярами.

Ключевые слова: оператор свертки, анизотропные пространства Бесова, анизотропные пространства Соболева, интерполяция.

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