DOI 10.31489/2019M2/48-55

UDC 517.95

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On a pseudo-Volterra nonhomogeneous integral equation

In this paper the issues of the solvability of a pseudo-Volterra nonhomogeneous integral equation of the second kind are studied. The solution to the corresponding homogeneous equation and the classes of the uniqueness of the solution are found in [1]. By replacing the right-hand side and the unknown function, the integral equation is reduced to an integral equation, the kernel of which is not «compressible». Using the Laplace transform, the obtained equation is reduced to an ordinary first-order differential equation (linear). Its solution is found. By using the solution of the homogeneous equation the form of a particular solution of the nonhomogeneous differential equation is defined (by the variation method of an arbitrary constant). By using the inverse Laplace transform, a particular solution of the pseudo-Volterra nonhomogeneous integral equation under study is obtained. The case of an nonhomogeneous integral equation with the value of the parameter $k = 1$ is considered and studied. Classes for the right side and the solution of the integral equation are indicated.

Keywords: pseudo-Volterra nonhomogeneous integral equation, class of essentially bounded functions, inverse Laplace transformation, residue.

Introduction

We research the solvability of the following nonhomogeneous pseudo-Volterra integral equation of the second kind

$$\nu(t) - \frac{a}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t\tau}\sqrt{t-\tau}} \exp\left(-\frac{t-\tau}{4a^2}\right) \cdot \nu(\tau)d\tau - \frac{1}{k\,a\sqrt{\pi}} \int_0^t \frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}} \exp\left(-\frac{t-\tau}{4a^2}\right) \cdot \nu(\tau)d\tau = f(t), \quad (1)$$

where $a,\ k$ are positive constants, $f(t)$ is the given function.

1. Reducing the equation (1) to a differential equation in images

Following the results of work [1] after replacements:

$$\frac{1}{\sqrt{t}} \exp\left(\frac{t}{4a^2}\right)\nu(t) = \nu_1(t), \quad \sqrt{t}\,\exp\left(\frac{t}{4a^2}\right) f(t) = f_1(t) \quad (2)$$

we get the following integral equation

$$t \cdot \nu_1(t) - \frac{a}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \cdot \nu_1(\tau)d\tau - \frac{1}{k\,a\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \cdot \tau\,\nu_1(\tau)\,d\tau = f_1(t). \quad (3)$$

Applying the Laplace transform the equation (3) transforms to the following differential equation

$$\left[\frac{1}{k\,a\sqrt{p}} - 1\right] \hat{\nu}_1'(p) - \frac{a}{2\sqrt{p}}\ \hat{\nu}_1(p) = \hat{f}_1(p). \quad (4)$$

The general solution to corresponding homogeneous (4) has the form

$$\hat{\nu}_{1,\,hom}(p) = C\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p} - \frac{1}{ka}\right)^{\frac{1}{k}}},$$

where $\mathcal{C} - const$.

Then the solution of the nonhomogeneous equation (4) is sought in the form

$$\hat{\nu}_1\,(p) = C(p)\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}. \tag{5}$$

Substituting function (5) into equation (4), we get

$$C(p) = -\int_p^{+\infty} \hat{f}_1(p)\cdot e^{a\sqrt{p}}\,\sqrt{p}\cdot\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}-1}\,dp + C. \tag{6}$$

After substituting (6) into (5), we can write out the partial solution of the nonhomogeneous equation (4) as

$$\hat{\nu}_{1,\,part}(p) = -\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_p^{+\infty}\left(\int_0^{+\infty} e^{-qt}f_1(t)dt\right)\cdot e^{a\sqrt{p}}\,\sqrt{q}\cdot\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1} dq.$$

Changing the order of integration, we obtain

$$\hat{\nu}_{1,\,part}(p) = -\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_0^{+\infty} f_1(t)\underbrace{\int_p^{+\infty} e^{-qt}e^{a\sqrt{q}}\,\sqrt{q}\cdot\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1} dq}_{I(p,t)}\,dt.$$

This way, we get

$$\hat{\nu}_{1,\,part}(p) = -\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_0^{+\infty} f_1(t)I(p,\,t)\,dt, \tag{7}$$

where

$$I(p,\,t) = \int_p^{+\infty}\sqrt{q}\cdot\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1} e^{-qt+a\sqrt{q}}dq. \tag{8}$$

We rewrite integral (8) in the form

$$I(p,\,t) = \underbrace{\int_p^{+\infty}\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}} e^{-qt+a\sqrt{q}}dq}_{J_{\frac{1}{k}}(p,\,t)} + \frac{1}{ka}\underbrace{\int_p^{+\infty}\left(\sqrt{q}-\frac{1}{ka}\right)^{\frac{1}{k}-1} e^{-qt+a\sqrt{q}}dq}_{J_{\frac{1}{k}-1}(p,\,t)}. \quad\Rightarrow$$

$$I(p,\,t) = J_n(p,\,t) + \frac{1}{ka}J_{n-1}(p,\,t), \tag{9}$$

where

$$J_{n-1}(p,\,t) = \int_p^{+\infty}\left(\sqrt{q}-\frac{1}{ka}\right)^{n-1} e^{-qt+a\sqrt{q}}dq = \left\|\begin{matrix}\sqrt{q}=x, & q=x^2;\\ dq=2xdx; & \sqrt{p}\le x<+\infty\end{matrix}\right\| =$$

$$= 2\int_{\sqrt{p}}^{+\infty} x^{-\frac{1}{ka}+\frac{1}{ka}}\left(x-\frac{1}{ka}\right)^{n-1} e^{-tx^2+ax}dx = 2\underbrace{\int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n} e^{-tx^2+ax}dx}_{A_n(p,t)} +$$

$$+\frac{2}{ka}\underbrace{\int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n-1} e^{-tx^2+ax}dx}_{A_{n-1}(p,t)} \quad\Rightarrow$$

$$J_{n-1}(p,\,t) = 2A_n(p,t) + \frac{2}{ka}A_{n-1}(p,t), \tag{10}$$

where

$$A_{n-1}(p,t) = \int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n-1}\exp(-tx^2+ax)dx =$$

$$= \exp\left(\frac{a^2}{4t}\right)\int_{\sqrt{p}}^{+\infty}\left(x-\frac{1}{ka}\right)^{n-1}\exp\left(-t\left(x-\frac{a}{2t}\right)^2\right)dx = \left\| \begin{array}{l} \lambda = x-\frac{a}{2t};\quad d\lambda = dx \\ \sqrt{p}-\frac{a}{2t} < \lambda < +\infty \end{array} \right\| =$$

$$= \exp\left(\frac{a^2}{4t}\right)\underbrace{\int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n-1}e^{-t\lambda^2}\,d\lambda}_{B_{n-1}(p,t)}$$

or

$$A_{n-1}(p,t) = \exp\left(\frac{a^2}{4t}\right)B_{n-1}(p,t), \tag{11}$$

where

$$B_r(p,t) = \int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{r}e^{-t\lambda^2}\,d\lambda.$$

We find the last integral when $r = n-1$:

$$B_{n-1}(p,t) = \int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n-1}e^{-t\lambda^2}\,d\lambda = \left\| \begin{array}{l} u = e^{-t\lambda^2};\ dv = \left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n-1}d\lambda \\ du = -2t\lambda e^{-t\lambda^2}d\lambda;\ v = \frac{1}{n}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n} \end{array} \right\| =$$

$$= \frac{1}{n}e^{-t\lambda^2}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n}\Bigg|_{\lambda=\sqrt{p}-\frac{a}{2t}}^{\lambda\to+\infty} + \frac{2t}{n}\int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^{n}\lambda e^{-t\lambda^2}\,d\lambda =$$

$$= -\frac{1}{n}\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)\left(\sqrt{p}-\frac{1}{ka}\right) + \frac{2t}{n}\left[B_{n+1}(p,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,t)\right].\Rightarrow$$

$$B_{n-1}(p,\,t) = -\frac{1}{n}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right) + \frac{2t}{n}\left[B_{n+1}(p,\,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,\,t)\right]. \tag{12}$$

We substitute the expression (12) into (11):

$$A_{n-1}(p,t) = \exp\left(\frac{a^2}{4t}\right)\left\{-\frac{1}{n}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$\left.+\frac{2t}{n}\left[B_{n+1}(p,\,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,\,t)\right]\right\}. \tag{13}$$

Then

$$A_n(p,\,t) = \exp\left(\frac{a^2}{4t}\right)\left\{-\frac{1}{n+1}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$\left.+\frac{2\,t}{n+1}\left[B_{n+2}(p,\,t)+\left(\frac{1}{ka}-\frac{a}{2t}\right)B_{n+1}(p,\,t)\right]\right\}. \tag{14}$$

Substituting the expressions (13) and (14) into (10) we get

$$J_{n-1}(p,\,t) = 2\exp\left(\frac{a^2}{4t}\right)\left\{-\frac{1}{n+1}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$+\frac{2\,t}{n+1}\left[B_{n+2}(p,\,t)+\left(\frac{1}{ka}-\frac{a}{2t}\right)B_{n+1}(p,\,t)\right]-\frac{1}{nka}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+$$

$$\left.+\frac{2t}{nka}\left[B_{n+1}(p,\,t)-\left(\frac{a}{2t}-\frac{1}{ka}\right)B_n(p,\,t)\right]\right\}.$$

Or

$$J_{n-1}(p,\,t) = 2\exp\left(\frac{a^2}{4t}\right)\left\{-\frac{nka+n+1}{nka(n+1)}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\right.$$

$$+\frac{2t}{n+1}B_{n+2}(p,t)+2t\left[\frac{1}{n+1}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{nka}\right]B_{n+1}(p,\,t)+\frac{2t}{n\,k\,a}\left(\frac{1}{ka}-\frac{a}{2t}\right)B_n(p,\,t)\Bigg\}. \tag{15}$$

Then

$$J_n(p,\,t)=2\exp\left(\frac{a^2}{4t}\right)\Bigg\{-\frac{(n+1)ka+n+2}{(n+1)ka(n+2)}\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+\frac{2t}{n+2}B_{n+3}(p,t)+$$

$$+2t\left[\frac{1}{n+2}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{(n+1)ka}\right]B_{n+2}(p,\,t)+\frac{2t}{(n+1)k\,a}\left(\frac{1}{ka}-\frac{a}{2t}\right)B_{n+1}(p,\,t)\Bigg\}. \tag{16}$$

Substituting the expressions (15) and (16) into (9) we get:

$$I(p,\,t)=2\exp\left(\frac{a^2}{4t}\right)\Bigg\{-\left(\frac{(n+1)ka+n+2}{(n+1)\,(n+2)ka}+\frac{nka+n+1}{n(n+1)\,(ka)^2}\right)\left(\sqrt{p}-\frac{1}{ka}\right)\exp\left(-t\left(\sqrt{p}-\frac{a}{2t}\right)^2\right)+$$

$$+\frac{2t}{n+2}B_{n+3}(p,\,t)+2t\left[\frac{1}{n+2}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{(n+1)ka}+\frac{1}{ka(n+1)}\right]B_{n+2}(p,\,t)+$$

$$+2t\left[\frac{2}{ka(n+1)}\left(\frac{1}{ka}-\frac{a}{2t}\right)+\frac{1}{n(ka)^2}\right]B_{n+1}(p,\,t)+\frac{2t}{n(ka)^2}\left(\frac{1}{ka}-\frac{a}{2t}\right)B_n(p,\,t)\Bigg\}. \tag{17}$$

Substituting the expression (17) into (7) we get $\hat{\nu}_{1,\;part}(p)$:

$$\hat{\nu}_{1,\;part}(p)=-\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\int_0^{+\infty}f_1(\theta)I(p,\,\theta)\,d\theta, \tag{18}$$

where the function $I(p,\,\theta)$ is defined by formula (17).

We rewrite (18) in the form

$$\hat{\nu}_{1,\;part}(p)=-\int_0^{+\infty}\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}\;I(p,\theta)\;f_1(\theta)\,d\theta. \tag{19}$$

We apply the inverse Laplace transform to (19)

$$\nu_{1,\;part}(t)=\frac{1}{2\pi\,i}\int_{c-i\infty}^{c+i\infty}\hat{\nu}_{1,\;part}(p)\,e^{pt}dp, \tag{20}$$

where the integration is performed along the line $Rep{=}c$, that is parallel to the imaginary axis and is shifted so that all singularities of function $\hat{\nu}_{1,part}$ lie on the left side of it.

Changing in (20) the order of integration θ and t, we get by virtue of the Cauchy residue theorem:

$$\nu_{1,\;part}(t)=-\int_0^{+\infty}f_1(\theta)\sum_{p_r}\underset{p=p_r}{res}\left\{\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}I(p,\;\theta)e^{pt}\right\}d\theta, \tag{21}$$

where p_r is a singular point of a function

$$G(p,\,\theta)=\frac{e^{-a\sqrt{p}+pt}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}I(p,\,\theta), \tag{22}$$

because by virtue of the formula

$$B_r(p,t)=\int_{\sqrt{p}-\frac{a}{2t}}^{+\infty}\left(\lambda+\frac{a}{2t}-\frac{1}{ka}\right)^r e^{-t\lambda^2}\,d\lambda.$$

and formula (17) we have

$$\lim_{p\to+\infty}\left\{\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p}-\frac{1}{ka}\right)^{\frac{1}{k}}}I(p,\ \theta)e^{pt}\right\}=0,$$

where ∞ is a removable singularity.

Obviously, the singular point of the function (22) (as a function of p) is the point $p=\dfrac{1}{k^2\,a^2}$. We will find the residue of the function $G(p,\,\theta)$ at this point.

We consider the case when $k=1$.

When $k=1$ from (8) we have:

$$I(p,\,\theta)=\int_{p}^{+\infty}\sqrt{q}\;e^{-q\theta+a\sqrt{q}}dq=2\int_{\sqrt{p}}^{+\infty}x^2e^{x^2\theta+ax}dx=2\exp\left(\frac{a^2}{4\theta}\right)\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}\left(\lambda+\frac{a}{2\theta}\right)^2e^{-\theta\lambda^2}d\lambda=$$

$$=2\exp\left(\frac{a^2}{4\theta}\right)\left[\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}\lambda^2e^{-\theta\lambda^2}d\lambda+\frac{a}{\theta}\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}\lambda\,e^{-\theta\lambda^2}d\lambda+\frac{a^2}{4\theta^2}\int_{\sqrt{p}-\frac{a}{2\theta}}^{+\infty}e^{-\theta\lambda^2}d\lambda\right]=$$

$$=\frac{1}{\theta}\exp\left(\frac{a^2}{4\theta}\right)\left[\left(\sqrt{p}+\frac{a}{2\theta}\right)\exp\left(-\theta\left(\sqrt{p}-\frac{a}{2\theta}\right)^2\right)+\left(1+\frac{a^2}{2\theta}\right)\operatorname{erfc}\left(\sqrt{p}-\frac{a}{2\theta}\right)\right]\Rightarrow$$

$$I(p,\,\theta)=\frac{1}{\theta}\exp\left(\frac{a^2}{4\theta}\right)\left[\left(\sqrt{p}+\frac{a}{2\theta}\right)\exp\left(-\theta\left(\sqrt{p}-\frac{a}{2\theta}\right)^2\right)+\left(1+\frac{a^2}{2\theta}\right)\operatorname{erfc}\left(\sqrt{p}-\frac{a}{2\theta}\right)\right].\tag{23}$$

Then at $p=\dfrac{1}{a^2}$ we obtain from (23):

$$I\left(\frac{1}{a^2};\ \theta\right)=\left(\frac{a}{2\theta^2}+\frac{1}{a\theta}\right)\exp\left(1-\frac{\theta}{a^2}\right)+\left(1+\frac{a^2}{2\theta}\right)\exp\left(\frac{a^2}{4\theta}\right)\operatorname{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right).\tag{24}$$

When $k=1$ ($p=\dfrac{1}{a^2}$ is a simple pole) for the function (22) we have in view of (24)

$$\underset{p=\frac{1}{a^2}}{res}\;G(p,\ \theta)=\frac{2}{a}\exp\left(\frac{t}{a^2}-1\right)\cdot I\left(\frac{1}{a^2};\ \theta\right)=$$

$$=\left(\frac{1}{\theta^2}+\frac{1}{a^2\theta}\right)\exp\left(\frac{t-\theta}{a^2}\right)+\left(\frac{2}{a}+\frac{a}{\theta}\right)\exp\left(\frac{t}{a^2}+\frac{a^2}{4\theta}-1\right)\operatorname{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right).$$

Then from (21) we get

$$\nu_{1,\,part}(t)=-\int_0^{+\infty}f_1(\theta)\left[\left(\frac{1}{\theta^2}+\frac{2}{a^2\theta}\right)\exp\left(\frac{t-\theta}{a^2}\right)+\right.$$

$$\left.+\left(\frac{2}{a}+\frac{a}{\theta}\right)\exp\left(\frac{t}{a^2}+\frac{a^2}{4\theta}-1\right)\operatorname{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right)\right]d\theta.\tag{25}$$

By virtue of replacements (2) from (25) we obtain a particular solution of the initial equation (1)

$$\nu_{part}(t)=-\sqrt{t}\exp\left(-\frac{3t}{4a^2}\right)\int_0^{+\infty}f(\theta)\left[\left(\frac{1}{\theta^{\frac{3}{2}}}+\frac{2}{a^2\sqrt{\theta}}\right)\exp\left(-\frac{3\theta}{4a^2}\right)+\right.$$

$$\left.+\left(\frac{2\sqrt{\theta}}{a}+\frac{a}{\sqrt{\theta}}\right)\exp\left(\frac{a^2}{4\theta}+\frac{\theta}{4a^2}\right)\operatorname{erfc}\left(\frac{1}{a}-\frac{a}{2\theta}\right)\right]d\theta.\tag{26}$$

Thus, the following theorem is proved.

Theorem. The integral equation

$$\nu(t)-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t\tau}\sqrt{t-\tau}}\exp\left(-\frac{t-\tau}{4a^2}\right)\cdot\nu(\tau)d\tau-\frac{1}{a\sqrt{\pi}}\int_0^t\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}}\exp\left(-\frac{t-\tau}{4a^2}\right)\cdot\nu(\tau)d\tau=f(t)$$

in the weight class of functions

$$\exp\left(-\frac{t}{4a^2}\right)\,\nu(\tau)\,\in L_\infty(0,\;+\infty)$$

at

$$\exp\left(-\frac{t}{4a^2}\right)\,f(t)\,\in L_\infty(0,\;+\infty)$$

has a solution defined by the formula (26).

Remark. Singular homogeneous integral equations were considered in works [2–4]. Their kernels were also «incompressible», but kernels had an another form. In this connection, the weight classes of the solution existence differ from the class of the solution existence for the equation considered in this work. We also note that boundary value problems for a spectrally loaded parabolic equation reduce to this kind of singular integral equations, when the load line moves according to the law $x = t$ [5–10] and problems for essentially loaded equation of heat conduction [11–15].

In works [16, 17] it is shown that the homogeneous Volterra integral equation of the second kind, to which the homogeneous boundary value problem of heat conduction in the degenerating domain is reduced, has a nonzero solution.

In works [18, 19] boundary value problems for heat equation in angular domains with special boundary conditions are studied. The problems are reduced to singular integral equations of Volterra type of the second kind, similar to the equation (1).

A similar kind of integral equation arises in solving the boundary value problems of heat conduction with heat generation, which describe the development of the one-dimensional unsteady heat processes with axial symmetry. More complex equations arise from the model that is based on the system of spherical heat equations in a domain with moving boundary and when studying the Stefan problem [20–23].

To find analytical solutions for classes of transfer problems, special methods or modification of known approaches are needed. Summary of the results accumulated in this area of the analytic theory of the thermal conductivity of solids is given in reviews [24, 25].

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Біртекті емес псевдо-Вольтерра интегралдық теңдеуі туралы

Мақалада екінші текті біртекті емес псевдо-Вольтерра теңдеуінің шешімі болады ма деген сұрақтар қарастырылды. Теңдеуге сәйкес біртекті теңдеудің шешімі және шешімінің жалғыздығының класы [1] жұмыста табылған. Берілген интегралдық теңдеудің оң жағы мен ізделінді функцияны ауыстыру арқылы, ядросы «қысылған» болмайтын интегралдық теңдеу түріне келді. Алынған теңдеу Лаплас түрлендіруінің көмегімен бірінші ретті қарапайым (сызықты) дифференцалдық теңдеуге алып келеді. Оның шешімі табылды. Біртекті теңдеудің шешімінің көмегімен біртекті емес дифференциалдық теңдеудің дербес шешімінің түрі анықталды (тұрақтыны вариациялау әдісімен). Лапластың кері түрлендіруін қолданып, зерттеліп отырған біртекті емес псевдо-Вольтерра интегралдық теңдеуінің дербес шешімі алынды. Біртекті емес интегралдық теңдеуінің параметрінің мәні $k = 1$ болған жағдайы зерттелді. Интегралдық теңдеудің оң жағы мен шешімі үшін кластары көрсетілді.

Кілт сөздер: біртекті емес псевдо-Вольтерра интегралдық теңдеуі, маңызды шектелген функциялар класы, Лапластың кері түрлендіруі, шегерім.

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Об одном псевдо-Вольтерровом неоднородном интегральном уравнении

В статье исследованы вопросы разрешимости псевдо-Вольтеррового неоднородного интегрального уравнения второго рода. Решение соответствующего однородного уравнения и классы единственности решения найдены в работе [1]. С помощью замен правой части и искомой функции интегральное уравнение сведено к интегральному уравнению, ядро которого не является «сжимаемым». С помощью преобразования Лапласа полученное уравнение сведено к обыкновенному дифференциальному уравнению первого порядка (линейному). Найдено его решение. С помощью решения однородного уравнения определен вид частного решения неоднородного дифференциального уравнения (методом вариации произвольной постоянной). Применением обратного преобразования Лапласа получено частное решение исследуемого псевдо-Вольтеррового неоднородного интегрального уравнения. Рассмотрен и исследован случай неоднородного интегрального уравнения при значении параметра $k = 1$. Указаны классы для правой части и решения интегрального уравнения.

Ключевые слова: псевдо-Вольтеррово неоднородное интегральное уравнение, класс существенно ограниченных функций, обратное преобразование Лапласа, вычет.