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Best trigonometric approximation and modulus of smoothness of functions in weighted grand Lebesgue spaces

In this work, first of all, $L_{\omega}^{p),\theta}(\mathbb{T})$ weighted grand Lebesgue spaces and Muckenhoupt weights is defined. The information about properties of these spaces is given. Let T_n be the trigonometric polynomial of best approximation. The approximation of the functions in grand Lebesgue spaces have been investigated by many authors. In this work the relation between fractional derivatives of a T_n trigonometric polynomial and the best approximation of the function is investigated in weighted grand Lebesgue spaces. In that regard, the necessary and sufficient condition is expressed in Theorem 1. In addition, in this work in weighted grand Lebesgue spaces a specific operator is defined. Later on, with the help of this operator the fractional modulus of smoothness of order r of function f is defined. Also, in this work, using the properties of modulus of smoothness of function, the relationship between the fractional modulus of smoothness of the function and n -th partial and de la Vallée-Poussin sums of its Fourier series in subspace of weighted grand Lebesgue spaces are studied. These results are expressed in Theorem 2.

Keywords: generalized grand Lebesgue spaces, fractional derivative, fractional moduli of smoothness, n -th partial sums, de la Vallée-Poussin sums, best approximation by trigonometric polynomials.

Introduction and the main results

Let \mathbb{T} denote the interval $[-\pi, \pi]$. We denote by $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of all measurable functions f on \mathbb{T} , that is, the space of all such functions for which

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p dx \right)^{1/p} < \infty.$$

A function ω is called a *weight* on \mathbb{T} if $\omega : \mathbb{T} \rightarrow [0, \infty]$ is measurable and $\omega^{-1}(\{0, \infty\})$ has measure zero (with respect to Lebesgue measure).

Let ω be a 2π periodic weight function. We denote by $L_{\omega}^p(\mathbb{T})$, $1 < p < \infty$, the weighted Lebesgue space of all measurable functions on \mathbb{T} for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \omega dx \right)^{1/p} < \infty.$$

We define a class $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$ of 2π periodic measurable functions on \mathbb{T} satisfying the condition

$$\sup_{0 < \varepsilon < p-1} \left\{ \frac{\varepsilon^{\theta}}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \omega(x) dx \right\}^{1/(p-\varepsilon)} < \infty.$$

The class $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$, is a Banach space with respect to the norm

$$\|f\|_{L_{\omega}^{p),\theta}(\mathbb{T})} := \sup_{0 < \varepsilon < p-1} \left\{ \varepsilon^{\theta} \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} \omega(x) dx \right\}^{1/(p-\varepsilon)}. \quad (1)$$

The class $L_\omega^{p,\theta}(\mathbb{T})$ with the norm (1) is called as the weighted generalized grand Lebesgue space. Note that non-weighted grand Lebesgue space $L^p(\mathbb{T})$ was introduced by Iwaniec and Sbordone [1]. Information about properties of these spaces can be found in [2–4]. The embeddings

$$L^p(\mathbb{T}) \subset L^p(\mathbb{T}) \subset L^{p-\varepsilon},$$

hold. According to [2] $L^p(\mathbb{T})$ is not dense in $L^p(\mathbb{T})$. Also, if $\theta_1 < \theta_2$ and $1 < p < \infty$, for weighted generalized grand Lebesgue space, the following relations hold:

$$L_\omega^p(\mathbb{T}) \subset L_\omega^{p,\theta_1}(\mathbb{T}) \subset L_\omega^{p,\theta_2}(\mathbb{T}) \subset L_\omega^{p-\varepsilon}(\mathbb{T}).$$

The closure of the space $L^p(\mathbb{T})$ by the norm of $L_\omega^{p,\theta}(\mathbb{T})$, $\theta > 0$, we denote by $\tilde{L}_\omega^{p,\theta}(\mathbb{T})$.

Let $1 < p < \infty$ and let $A_p(\mathbb{T})$ be the collection of all weights on \mathbb{T} satisfying the condition

$$\sup_I \left(\frac{1}{|I|} \int_I \omega(x)^p dx \right)^{1/p} \left(\frac{1}{|I|} \int_I [\omega(x)]^{-1/(p-1)} dx \right)^{p-1} < \infty, \quad (2)$$

where the supremum is taken over all intervals I with length $|I| \leq 2\pi$. The condition (2) is called the *Muckenhoupt - A_p* condition [5] and the weight functions which belong to $A_p(\mathbb{T})$, ($1 < p < \infty$), are called as the *Muckenhoupt weights*.

Suppose that $f \in L_\omega^{p,\theta}$. We define the operator by

$$A_h f(x) := \frac{1}{h} \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t) dt, \quad x \in \mathbb{T} \quad 0 < h \leq 1.$$

Note that $0 < p < \infty$, $\theta > 0$ and $\omega \in A_p$ then the operator A_h is bounded in $L_\omega^{p,\theta}$. For $f \in L_\omega^{p,\theta}$, we define

$$\begin{aligned} \sigma_h^r f(x) &:= (I - A_h)^r f(x) = \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1) \Gamma(r-k+1)} (A_h)^k f, \quad x, h \in \mathbb{T}, \quad 0 \leq r, \end{aligned}$$

where I is the identity operator and Γ is gamma function.

Let $\omega \in A_p$ and $f \in L_\omega^{p,\theta}$. If $0 \leq r$ we can define the *fractional modules of smoothness of order r* of f as

$$\Omega_r(f, \delta)_{p,\theta,\omega} := \sup_{0 \leq h_i, t \leq \delta} \left\| \prod_{i=1}^{[r]} (I - A_{h_i}) \sigma_t^{\{r\}} f \right\|_{p,\theta,\omega}, \quad \delta \geq 0,$$

where $[r]$ denotes the integer part of the real number r and $\{r\} := r - [r]$. – Note that $\Omega_0(f, \delta)_{p,\theta,\omega} := \|f\|_{p,\theta,\omega}$ and $\prod_{i=1}^0 (I - A_{h_i}) \sigma_t^{\{r\}} f := \sigma_t^r f$ for $0 < r < 1$. The modulus of smoothness $\Omega_r(f, \delta)_{p,\theta,\omega}$, $r \in \mathbb{R}^+$, is a nondecreasing, nonnegative function of δ , and

$$\begin{aligned} \Omega_r(f + g, \delta)_{p,\theta,\omega} &\leq \Omega_r(f, \delta)_{p,\theta,\omega} + \Omega_r(g, \delta)_{p,\theta,\omega}, \\ \lim_{\delta \rightarrow 0} \Omega_r(f, \delta)_{p,\theta,\omega} &= 0 \end{aligned}$$

for $f, g \in L_\omega^{p,\theta}$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(f, x), \quad A_k(f, x) := a_k(f) \cos kx + b_k(f) \sin kx \quad (3)$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f . The n th partial sums and de la Vallée-Poussin sums of the series (3) are defined, respectively, as:

$$S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^n A_k(f, x),$$

$$V_n(f, x) : = \frac{1}{n} \sum_{\nu=n}^{2n-1} S_\nu(f, x).$$

We denote by $E_n(f)_{p, \theta, \omega}$, ($n = 0, 1, 2, \dots$) the best approximation of $f \in L_{2\pi}^{p(\cdot)}$ by trigonometric polynomials of degree not exceeding n , i.e.

$$E_n(f)_{p, \theta, \omega} := \inf \left\{ \|f - T_n\|_{p, \theta, \omega} : T_n \in \Pi_n \right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n .

We use the relation $\alpha_n = O(\beta_n)$, $n = 1, 2, \dots$, that is, there exists a constant $C > 0$ such as $\alpha_n \leq C\beta_n$, $n = 1, 2, \dots$

The approximation problems of the functions by trigonometric polynomials in grand Lebesgue spaces have been investigated by several authors [6–13].

In the present paper, in weighted generalized grand Lebesgue spaces we investigate the relation between derivatives of a polynomial of best approximation and the best approximation of the function. In addition, relationship between fractional modulus of smoothness of the function and n th partial and de la Vallée-Poussin sums of its Fourier series in subspace of weighted grand Lebesgue spaces are studied. Similar results in different spaces have been investigated in [14–34], and [5]. Note that, in the proof of the main results we use the method as in the proof of [27, 28].

Our main results are the following:

Theorem 1. Let $T_n(f) \in \Pi_n$ be the polynomial of best approximation to f , let $r, \alpha \in \mathbb{R}^+$. In order that

$$\left\| T_n^{(r)}(f) \right\|_{p, \theta, \omega} = O(n^{r-\alpha}) \quad (r > \alpha > 0)$$

it is necessary and sufficient that

$$E_n(f)_{p, \theta, \omega} = O(n^{-\alpha}).$$

Theorem 2. Let $1 < p < \infty$, $\theta > 0$, $r \in \mathbb{R}^+$ and $\omega \in A_p$. If $f \in \tilde{L}_\omega^{p, \theta}(\mathbb{T})$, then

1.

$$c_4 \Omega_r\left(f, \frac{1}{n}\right)_{p, \theta, \omega} \leq n^{-2r} \left\| V_n^{(2r)}(f, \cdot) \right\|_{p, \theta, \omega} + \|f(x) - V_n(f, \cdot)\|_{p, \theta, \omega} \leq c_5 \Omega_r\left(f, \frac{1}{n}\right)_{p, \theta, \omega}, \quad (4)$$

where the constants c_4 and c_5 are dependent on p and r .

2.

$$c_6 \Omega_r\left(f, \frac{1}{n}\right)_{p, \theta, \omega} \leq n^{-2r} \left\| S_n^{(2r)}(f, \cdot) \right\|_{p, \theta, \omega} + \|f(x) - S_n(f, \cdot)\|_{p, \theta, \omega} \leq c_7 \Omega_r\left(f, \frac{1}{n}\right)_{p, \theta, \omega}, \quad (5)$$

where the constants c_6 and c_7 are dependent on p and r .

Proofs of the main results

Proof of Theorem 1. Let us assume that

$$E_n(f)_{p, \theta, \omega} = \|f - T_n(f)\|_{p, \theta, \omega} = O(n^{-\alpha}), \quad (\alpha > 0). \quad (6)$$

is satisfied. We can write

$$T_n^{(r)}(x) = T_0^{(r)}(x) + \sum_{\nu=0}^{n-1} \left\{ T_{\nu+1}^{(r)}(x) - T_{\nu}^{(r)}(x) \right\}. \quad (7)$$

Using Bernstein inequality for the spaces $\tilde{L}_{\omega}^{(p),\theta}(\mathbb{T})$ in [35] we have

$$\left\| T_n^{(r)}(f) \right\|_{p,\theta,\omega} \leq c_8 n^r \|T_n(f)\|_{p,\theta,\omega}.$$

From (6), (7) and the last relation we conclude that

$$\left\| T_n^{(r)}(f) \right\|_{p,\theta,\omega} \leq c_9 c_{10} n^r n^{-\alpha} \leq c_{11} n^{r-\alpha}.$$

Now we suppose that

$$\left\| T_n^{(r)}(f) \right\|_{p,\theta,\omega} = O(n^{r-\alpha}). \quad (8)$$

Use of [9] and (8) leads to

$$\begin{aligned} \|T_{2n}(f) - T_n(T_{2n}(f))\|_{p,\theta,\omega} &\leq \|f - T_{2n}(f)\|_{p,\theta,\omega} + \|f - T_n(T_{2n}(f))\|_{p,\theta,\omega} \leq \\ &\leq c_{12} n^{-r} \left\| T_n^{(r)}(f) \right\|_{p,\theta,\omega} \leq c_{13} n^{-\alpha}. \end{aligned} \quad (9)$$

On the other hand, since $T_n(T_{2n}(f))$ is a polynomial of order n the following inequality holds:

$$\begin{aligned} \|T_{2n}(f) - T_n(T_{2n}(f))\|_{p,\theta,\omega} &= \|f - T_n(T_{2n}(f)) - (f - T_{2n}(f))\|_{p,\theta,\omega} \geq \\ &\geq \|f - T_n(T_{2n}(f))\|_{p,\theta,\omega} - \|f - T_{2n}(f)\|_{p,\theta,\omega} \geq \\ &\geq E_n(f)_{p,\theta,\omega} - E_{2n}(f)_{p,\theta,\omega} \geq 0. \end{aligned} \quad (10)$$

Use of (9) and (10) gives us

$$0 \leq E_n(f)_{p,\theta,\omega} - E_{2n}(f)_{p,\theta,\omega} \leq c_{14} n^{-\alpha}. \quad (11)$$

Condition $E_n(f)_{p,\theta,\omega} \rightarrow 0$ is satisfied. Therefore, from the inequality (11) we have

$$\sum_{k=n_0}^{\infty} \{E_{2^k}(f)_{p,\theta,\omega} - E_{2^{k+1}}(f)_{p,\theta,\omega}\} \leq c_{15} \sum_{k=n_0}^{\infty} 2^{-k\alpha}.$$

Thus,

$$E_{2^{n_0}}(f)_{p,\theta,\omega} \leq c_{16} 2^{-n_0\alpha}. \quad (12)$$

By (12) we conclude that $E_n(f)_{p,\theta,\omega} \leq c_{15}(n^{-\alpha})$.

This completes the proof Theorem 1.

Proof of Theorem 2. By [9] the inequality

$$\Omega_r\left(T_n, \frac{1}{n}\right)_{p,\theta,\omega} \leq c_{17}(p,r)n^{-2r} \left\| T_n^{(2r)} \right\|_{p,\theta,\omega}, \quad (13)$$

holds, where $T_n \in \Pi_n$. On the other hand, using the properties of modulus of smoothness $\Omega_r(f, \frac{1}{n})_{p,\theta,\omega}$ and (13), we find

$$\begin{aligned} \Omega_r\left(f, \frac{1}{n}\right)_{p,\theta,\omega} &\leq \left(\Omega_r\left(f - T_n, \frac{1}{n}\right)_{p,\theta,\omega} + \Omega_r\left(T_n, \frac{1}{n}\right)_{p,\theta,\omega} \right) \leq \\ &\leq c_{18}(p,r) \left(\|f - T_n\|_{p,\theta,\omega} + n^{-2r} \left\| T_n^{(2r)} \right\|_{p,\theta,\omega} \right). \end{aligned}$$

Now we estimate the modulus of smoothness $\Omega_r(f, \cdot)_{p, \theta, \omega}$ from below. According to reference [10] the following inequalities hold:

$$E_n(f)_{p, \theta, \omega} \leq c_{19}(p, r) \Omega_r \left(f, \frac{2\pi}{n+1} \right)_{p, \theta, \omega}; \quad (14)$$

$$n^{-2r} \left\| T_n^{(2r)} \right\|_{p, \theta, \omega} \leq c_{20}(p, r) \Omega_r \left(f, \frac{2\pi}{n+1} \right)_{p, \theta, \omega}. \quad (15)$$

Let $V_n(f, x)$ be de la Vallée-Poussin sums of the series (3) and let $T_n^* \in \Pi_n$ be the polynomial of best approximation to f in $\tilde{L}_\omega^{p, \theta}(\mathbb{T})$, that is $\|f - T_n^*\|_{p, \theta, \omega} = E_n(f)_{p, \theta, \omega}$. Then we get

$$\begin{aligned} \|f - V_n(f, \cdot)\|_{p, \theta, \omega} &\leq \|f - T_n^*\|_{p, \theta, \omega} + \|T_n^* - V_n(f, \cdot)\|_{p, \theta, \omega} \leq \\ &\leq c_{21}(p) E_n(f)_{p, \theta, \omega} + \|V_n(T_n^* - f, \cdot)\|_{p, \theta, \omega} \leq \\ &\leq c_{22}(p) E_n(f)_{p, \theta, \omega}. \end{aligned} \quad (16)$$

Using (14), (15) and (16) we have

$$\begin{aligned} n^{-2r} \left\| V_n^{(2r)}(f, \cdot) \right\|_{p, \theta, \omega} + \|f - V_n(f, \cdot)\|_{p, \theta, \omega} &\leq \\ &\leq c_{23}(p, r) \left(\Omega_r(V_n, \frac{1}{n})_{p, \theta, \omega} + E_n(f)_{p, \theta, \omega} \right) \leq \\ &\leq c_{24}(p, r) \left(\Omega_r(f, \frac{1}{n})_{p, \theta, \omega} + \Omega_r(f - V_n, \frac{1}{n})_{p, \theta, \omega} + E_n(f)_{p, \theta, \omega} \right) \leq \\ &\leq c_{25}(p, r) \Omega_r(f, \frac{1}{n})_{p, \theta, \omega}. \end{aligned}$$

which completes the estimation (4) of Theorem 2.

Let T_n be the best approximation polynomial for f , i.e.,

$$E_n(f)_{p, \theta, \omega} = \|f - T_n\|_{p, \theta, \omega}.$$

By [9], Theorem 5; [10], Theorem 2.1 there exists a constant $c_{25}(p)$ such as

$$\|f - S_n(f, \cdot)\|_{p, \theta, \omega} \leq \|f - T_n\|_{p, \theta, \omega} + \|S_n(T_n - f)\|_{p, \theta, \omega} \leq c_{26}(p) E_n(f)_{p, \theta, \omega}. \quad (17)$$

Using inequality (17) and the scheme of proof of the estimation (4) we have the estimate (5).

Proof of Theorem 2 is completed.

References

- 1 Iwaniec, T., & Sbordone, C. (1992). On the integrability of the Jacobian under minimal hypotheses. *Arch. Ration. Mech. Anal.* 119, 2, 129–143.
- 2 D'onofrio, L., Sbordone, C., & Schiattarella, R. (2013). Grand Sobolev spaces and their application in geometric function theory and PDEs. *J. Fixed Point Theory Appl.* 13, 309–340.
- 3 Iwaniec, T., & Sbordone, C. (1998). Riesz transforms and elliptic PDEs with VMO coefficients. *J. Anal. Math.* 74, 183–212.
- 4 Kokilshvili, V. (2011). Boundedness criteria for singular integrals in weighted grand Lebesgue spaces. *J. Math. Sci.*, 170, 20–33.
- 5 Ky, N.X. (1997). Moduli of mean smoothness and approximation with A_p -weights. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 40, 37–38.
- 6 Danelia, N., & Kokilashvili, V. (2012). On the approximation of periodic functions within the frame of grand Lebesgue spaces. *Bull. Georgian Nat. Acad. Sci.* 6, 2, 11–16.

- 7 Danelia, N.,V., & Kokilashvili, V. (2013). Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue spaces. *Bull. Georg. Natl. Acad. Sci.*, 7, 1, 11–15.
- 8 Danelia, N.,V., & Kokilashvili, V. (2014). Approximation of periodic functions in grand variable exponent Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* 164, 100–103.
- 9 Danelia, N., Kokilashvili V., & Tsanava, Ts. (2014). Some approximation results in subspace of weighted grand Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* 164, 104–108.
- 10 Danelia, N., & Kokilashvili, V. (2016). Approximation by trigonometric polynomials in the framework of variable exponent grand Lebesgue spaces. *Georgian Math. J.* 23, 1, 43–53.
- 11 Israfilov, D.M., & Testici, A. (2016). Approximation in weighted generalized grand Lebesgue spaces. *Colloquim Mathematicum*, 143, 1, 113–126.
- 12 Israfilov, D.M., & Testici, A. (2017). Approximation in weighted generalized grand Smirnov classes, *Studia Scientiarum Mathematicarum Hungarica*, 54 (4), 471–488.
- 13 Jafarov, S.Z. (2016). Approximation by trigonometric polynomials in subspace of variable exponent grand Lebesgue spaces. *Global J. Math.* 8, 2, 836–843.
- 14 Akgün, R. (2010). Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* 152, 1–18.
- 15 Akgün, R. (2011). Polynomial approximation in weighted Lebesgue spaces. *East. J. Approx.* 17, 3, 253–266.
- 16 Akgün, A., & Kokilashvili, V. (2011). The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces. *Georgian Math. J.* 18, 3, 399–423.
- 17 Akgün, R. (2011). Approximation in weighted Orlicz spaces. *Math. Slovaca* 61, 4, 601–618.
- 18 Akgün, R. (2015). Realization and characterization of modulus of smoothness in weighted Lebesgue spaces. *St. Petersburg Math. J.* 26, 5, 741–756.
- 19 Butzer, P.L., & Pawelke, S. (1967). Ableitungen von trigonometrischen Approximationsprozessen. *Acta Sci. Math. (Szeged)*, Vol. 28, 173–183.
- 20 DeVore, R.A., & Lorentz, G.G. (1993). *Constructive approximation*. Berlin, Springer-Verlag Berlin Heidelberg.
- 21 Guven, A., & Israfilov, D.M. (2009). Approximation by Means of Fourier trigonometric series in weighted Orlicz spaces. *Adv. Stud. Contemp. Math. (Kyundshang)*, 19, 2, 283–295.
- 22 Guven, A., & Israfilov, D.M. (2010). Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$. *J. Math. Inequal.* 4, 285–290.
- 23 Israfilov, D.M., & Ali Guven, A. (2006). Approximation by trigonometric polynomials in weighted Orlicz spaces. *Studia Mathematica*, 174 (2), 147–168.
- 24 Israfilov, D.M., Kokilashvili V., & Samko, S.G. (2007). Approximation in weighted Lebesgue spaces and Smirnov spaces with variable exponent. *Proc. A. Razmadze Math. Inst.* 143, 25–35.
- 25 Jafarov, S.Z. (2015). Linear methods of summing Fourier series and approximation in weighted variable exponent Lebesgue spaces. *Ukr. Math. J.* 66, 10, 1509–1518.
- 26 Kokilashvili, V.M., & Samko, S.G. (2009). A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* 151, 134–138.
- 27 Sunouchi, G. (1968). Derivatives of a polynomial of best approximation. *Jahresbericht d. DMV*, 70, 165–166.
- 28 Simonov, B.V., & Tikhonov, S.Yu. (2008). Embedding theorems in the constructive theory of approximations. *Sbornik: Mathematics*, 199 (9), 1367–1407.
- 29 Stechkin, S.B. (1951). On the order of the best approximations of continuous functions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 15, 3, 219–242.
- 30 Sharapudinov, I.I. (2013). Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials. *Izvestiya: Mathematics*, 77 (2), 407–434.
- 31 Sharapudinov, I.I. (2014). Approximation of functions in variable exponent Lebesgue and Sobolev spaces. *Sbornik: Mathematics*, 205 (1–2), 291–30.

- 32 Timan, A.F. (1963). *Theory of approximation of functions of a real variable*. New York: Pergamon Press and MacMillan.
- 33 Tsanova, T., & Kokilashvili, V. (2012) Some notes on the majorants of Fourier partial sums in new function spaces. *J. Tech. Sci. Technologists* 1, 29–31.
- 34 Zygmund, A. (1959). *Trigonometric Series, Parts I and II*. Cambridge: Cambridge University Press.
- 35 Kokilashvili, V.M., & Meskhi, A. (2015). On weighted Bernstein type inequality in grand variable exponent Lebesgue spaces. *Math. Ineq. and Appl.* 18, 3, 991–1002.

С.З. Джафаров

Ең жақын тригонометриялық жуықтау және Лебегтің салмақтық гранд-кеңістіктеріндегі функцияның тегістігінің модулі

Мақалада бірінші кезекте Лебегтің салмақтық гранд-кеңістіктері және Макенхаупт $L_{\omega}^{p,\theta}(\mathbb{T})$ салмақтары анықталды. Осы кеңістіктердің қасиеттері жайлы ақпарат берілді. T_n ең жақын жуықтаудың тригонометриялық полиномы болсын. Лебегтің гранд-кеңістіктеріндегі функцияны аппроксимациялау көптеген авторлармен зерттелді. Бұл жұмыста тригонометриялық полиномдағы T_n бөлшек туынды мен Лебегтің салмақтық гранд-кеңістіктеріндегі ең жақын жуықтау арасындағы байланыс қарастырылды. Осыған байланысты қажетті және жеткілікті шарттар 1-теоремада келтірілді. Автор Лебегтің салмақтық гранд-кеңістіктерінде нақты операторды анықтаған. Кейінірек бұл оператордың көмегімен f функциясының r ретті тегістігінің бөлшекті модульдері анықталды. Сонымен қоса бұл жұмыста функцияның тегістігінің модулінің қасиеттерін қолдана отырып, тегістіктің бөлшекті модулі мен Лебегтің салмақтық гранд-кеңістігінің ішкі кеңістігіндегі Фурье қатарының де Валле-Пуссеннің $n - th$ дербес қосындылары арасындағы өзара байланыс қарастырылды. Бұл нәтижелер 2-теоремада келтірілген.

Кілт сөздер: Лебегтің жалпыланған гранд-кеңістіктері, бөлшекті туынды, тегістіктің бөлшекті модульдері, $n - th$ дербес қосындылары, де Валле-Пуссен қосындылары, тригонометриялық полиномдармен ең жақын жуықтау.

С.З. Джафаров

Наилучшее тригонометрическое приближение и модуль гладкости функций в весовых гранд-пространствах Лебега

В статье, в первую очередь, определены весовые гранд-пространства Лебега и веса Макенхаупта $L_{\omega}^{p,\theta}(\mathbb{T})$. Дана информация о свойствах этих пространств. Пусть T_n будет тригонометрическим полиномом наилучшего приближения. Аппроксимация функций в гранд-пространствах Лебега исследовалась многими авторами. В этой работе изучена связь между дробными производными T_n тригонометрического полинома и наилучшим приближением функции в весовых гранд-пространствах Лебега. В связи с этим необходимое и достаточное условие выражено в теореме 1. Кроме того, в этой работе в весовых гранд-пространствах Лебега определен конкретный оператор. Позже с помощью этого оператора будут определены дробные модули гладкости порядка r функции f . Также, используя свойства модуля гладкости функции, автором изучена взаимосвязь между дробным модулем гладкости и $n - th$ частичными суммами де Валле-Пуссена ряда Фурье в подпространстве весового гранд-пространства Лебега. Эти результаты выражены в теореме 2.

Ключевые слова: обобщенные гранд-пространства Лебега, дробная производная, дробные модули гладкости, $n - th$ частичные суммы, суммы де Валле-Пуссена, наилучшее приближение тригонометрическими полиномами.