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A.Sh. Akysh (Akishev)

Almaty, Kazakhstan
(E-mail: akysh41@mail.ru)

Splitting method and the existence of a strong solution of the Navier-Stokes equations[1]

In the author's article from the previous issue of the journal from the properties of the ONS solutions the relation between pressure and module square of velocity vector is set. Based on which the uniqueness of the weak and existence of strong solutions of the problem for three-dimensional equations of Navier-Stokes as a whole over time are proved. The result is a contribution to a qualitative mathematical theory of the Navier-Stokes equations. However, one of the actual problems in the theory of equations of Navier-Stokes is the choice of the mathematical method for proofs of the existence of a theorem. In the work splitting method is chosen to solve the Navier-Stokes equations. The rationale of this method is given. The compactness of the solution sequence is showed, thus the existence of strong solutions of the problem for three-dimensional Navier-Stokes equations as a whole over time is proved.

Keywords: the Navier-Stokes equations, splitting method for the Navier-Stokes equations, compactness, the existence of strong solutions, determination algorithm of strong solutions.

0.1 Problem statement and splitting method

In [1, 2] the initial-boundary value problem for nonlinear equations of Navier-Stokes relatively to the velocity vector $\mathbf{U} = (U_1, U_2, U_3) \in \mathring{\mathbf{J}}(Q)$ and the pressure P in domain $Q = (0, T] \times \Omega$ is reduced to

$$\frac{\partial \mathbf{U}}{\partial t} - \mu\Delta\mathbf{U} + (\mathbf{U}, \nabla)\mathbf{U} - \nabla\left|\mathbf{U}\right|^2 = \mathbf{f}(t, \mathbf{x}), \tag{1a}$$

$$\mathbf{U}(0, \mathbf{x}) = \mathbf{\Phi}(\mathbf{x}),\ \mathbf{U}(t, \mathbf{x})\big|_{\mathbf{x}\in\partial\Omega} = 0, \tag{1b}$$

where $t \in (0, T], \forall T < \infty$; $\mathbf{x} \in \Omega$, $\Omega \subset R_3$, $\partial\Omega$ — is the boundary of Ω, $\mathbf{x} \in \Omega \subset R_3$; Ω is a convex domain $\mathring{\mathbf{J}}(\Omega)$ - space solenoidal vectors; $\mathbf{L}_\infty(Q)$ — is the subspace of $\mathbf{C}(\bar{Q})$. $W^k_{p,0}(\Omega)$ is Sobolev space functions equal to zero on $\partial\Omega$;

The input data $\mathbf{f}$ and $\mathbf{\Phi}$ of the problem (1) meet the requirements:

$$\textbf{i)}\ \mathbf{f}(t, \mathbf{x}) \in \mathbf{L}_\infty(0, T; \mathbf{L}_p(\Omega)) \cap \mathring{\mathbf{J}}(Q); \quad \textbf{ii)}\ \mathbf{\Phi}(\mathbf{x}) \in \mathbf{L}_p(\Omega) \cap \mathbf{W}^1_{2,0}(\Omega) \cap \mathring{\mathbf{J}}(\Omega),\ \forall p.$$

[1]The work was done on the personal initiative of the author, since there was no financial support.

Further, we use the Holder inequalities

$$\left|\int\limits_{\Omega} UV\,\mathbf{dx}\right| \le \left(\int\limits_{\Omega} |U|^p\,\mathbf{dx}\right)^{\frac{1}{p}} \left(\int\limits_{\Omega} |V|^q\,\mathbf{dx}\right)^{\frac{1}{q}} \tag{2}$$

in addition, the integration by parts formula

$$\int\limits_{\Omega} V\Delta U\,\mathbf{dx} = -\int\limits_{\Omega} \nabla V\,\nabla U\,\mathbf{dx} + \int\limits_{\partial\Omega} V\frac{\partial U}{\partial \mathbf{n}}\,\mathbf{dx}. \tag{3}$$

To solve the problem (1) we use the splitting method. Let known vector-function be an approximation $\{\mathbf{u}^n\}$ in the moment $n\tau$, $0<\tau-$ step, then vector functions $\{\mathbf{u}^{n+1/2}\}$, $\{\mathbf{u}^{n+1}\}$, $n=0,1,\cdots,M$; $T=M\tau<\infty$ determined by the decisions of the following subsystems:

$$\frac{\mathbf{U}^{n+1/2}-\mathbf{U}^n}{\tau} + (\mathbf{U}^n,\nabla)\mathbf{U}^{n+1/2} - \nabla|\mathbf{U}^{n+1/2}|^2 = \mathbf{f}^n, \tag{4}$$

with initial

$$\mathbf{u}^0(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}) \wedge \mathbf{\Phi}(\mathbf{x})\big|_{\partial\Omega} = 0 \tag{5}$$

and

$$\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n+1/2}}{\tau} - \mu\Delta\mathbf{U}^{n+1} = 0, \tag{6}$$

boundary conditions

$$\mathbf{U}^{n+1}\big|_{\partial\Omega} = 0, \quad n=0,1,\cdots,M-1. \tag{7}$$

Lemma. For solving the splitting method scheme(4)-(7) fair estimates:

$$\|\mathbf{U}^{n+1/2}\|_{L_p(\Omega)} \le \|\mathbf{U}^n\|_{L_p(\Omega)} + \tau\|\mathbf{f}^n\|_{L_p(\Omega)}; \tag{8}$$

$$\max_{0\le n\le M-1} \|\mathbf{U}^{n+1}\|_{L_p(\Omega)} \le \|\mathbf{\Phi}\|_{L_p(\Omega)} + T\max_{0\le n<M}\|\mathbf{f}^n\|_{L_p(\Omega)},$$
$$\forall p=2k,\ k\in N,\ n=0,1,\cdots,M-1. \tag{9}$$

Proof. Multiply the scalar equation (4) on a function vector

$$p(E^{n+1/2})^{p-1}\mathbf{U}^{n+1/2},$$

we integrate work over the domain Ω and let us use identity $E^p = \frac{1}{2^p}|\mathbf{U}|^{2p}$, then similarly, both of works (see [1], inequalities (10)–(14)) find

$$2\int\limits_{\Omega}(E^{n+1/2})^p d\mathbf{x} + \tau\int\limits_{\Omega}\mathbf{U}^n\nabla(E^{n+1/2})^p d\mathbf{x} - 2\tau\int\limits_{\Omega}\mathbf{U}^{n+1/2}\nabla(E^{n+1/2})^p d\mathbf{x} =$$
$$= p\int\limits_{\Omega}(E^{n+1/2})^{p-1}\mathbf{U}^{n+1/2}\big(\mathbf{U}^n+\tau\mathbf{f}^n\big)d\mathbf{x}. \tag{10}$$

The second term from the left side (10) is transformed with integration in parts[2], then the right-hand side is estimated by the inequality Holder:

$$\tau\int\limits_{\Omega}\mathbf{U}^n\nabla\left(E^{n+1/2}\right)^p d\mathbf{x} = -\tau\int\limits_{\Omega} div\mathbf{U}^n\left(E^{n+1/2}\right)^p d\mathbf{x} + \tau\int\limits_{\partial\Omega}\mathbf{U}^n\mathbf{n}\left(E^{n+1/2}\right)^p d\mathbf{x} = 0; \tag{11}$$

$$-2\tau\int\limits_{\Omega}\mathbf{U}^{n+1/2}\nabla\left(E^{n+1/2}\right)^p d\mathbf{x} = 2\tau\int\limits_{\Omega} div\mathbf{U}^{n+1/2}\left(E^{n+1/2}\right)^p d\mathbf{x} - 2\tau\int\limits_{\partial\Omega}\mathbf{U}^{n+1/2}\mathbf{n}\left(E^{n+1/2}\right)^p d\mathbf{x} = 0; \tag{12}$$

[2] Of(6) follows $div\mathbf{U}^{n+1/2}=0$, consequently in equality (12) $\mathbf{U}^{n+1/2}\mathbf{n}=0$ [3; 46].

$$p\int\limits_{\Omega}\left(E^{n+1/2}\right)^{p-1}\mathbf{U}^{n+1/2}\mathbf{U}^{n}d\mathbf{x}\le\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1/2}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}};\qquad(13)$$

$$p\tau\int\limits_{\Omega}\left(E^{n+1/2}\right)^{p-1}\mathbf{U}^{n+1/2}\mathbf{f}^{n}d\mathbf{x}\le\tau\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1/2}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}.\qquad(14)$$

Now from the identity (10), using the relation (11)–(14), we arrive at the inequality

$$\frac{p}{2^{p-1}}\int\limits_{\Omega}|\mathbf{U}^{n+1/2}|^{2p}d\mathbf{x}\le$$

$$\le\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}|\mathbf{U}^{n+1/2}|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\left(\int\limits_{\Omega}|\mathbf{U}^{n}|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}+\tau\left(\int\limits_{\Omega}|\mathbf{f}^{n}|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}\right).$$

Where there is the estimate for the fractional step $n+1/2$ of the splitting method, that is (8) of the lemma 1.

$$\Big(\int\limits_{\Omega}|\mathbf{U}^{n+1/2}|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}\le\Big(\int\limits_{\Omega}|\mathbf{U}^{n}|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}+\tau\Big(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}\quad\text{or}$$

$$\|\mathbf{U}^{n+1/2}\|_{L_p(\Omega)}\le\|\mathbf{U}^{n}\|_{L_p(\Omega)}+\tau\|\mathbf{f}^{n}\|_{L_p(\Omega)},\ \forall p=2k,\ k\in N.$$

To obtain the estimate (9) for the whole step, multiply the equation (6) by vector function $p(E^{n+1})^{p-1}\mathbf{U}^{n+1}$ and integrate the result over Ω

$$2\int\limits_{\Omega}\left(E^{n+1}\right)^{p}d\mathbf{x}-\tau p\mu\int\limits_{\Omega}\left(\Delta\mathbf{U}^{n+1},\mathbf{U}^{n+1}\right)\left(E^{n+1}\right)^{p-1}d\mathbf{x}-$$

$$-\tau p\int\limits_{\Omega}\left(\nabla|\mathbf{U}^{n+1}|^{2},\mathbf{U}^{n+1}\right)\left(E^{n+1}\right)^{p-1}d\mathbf{x}=p\int\limits_{\Omega}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}\left(E^{n+1}\right)^{p-1}d\mathbf{x}+$$

$$+\tau p\int\limits_{\Omega}\mathbf{U}^{n+1}\mathbf{f}^{n}(E^{n+1})^{p-1}d\mathbf{x}.\qquad(15)$$

For the second term on the left side (15) we find

$$-\tau p\mu\int\limits_{\Omega}(\Delta\mathbf{U}^{n+1},\mathbf{U}^{n+1})(E^{n+1})^{p-1}d\mathbf{x}=\tau p\mu\int\limits_{\Omega}(E^{n+1})^{p-1}\sum_{\alpha=1}^{3}(\nabla U_{\alpha}^{n+1})^{2}d\mathbf{x}+$$

$$+\tau p(p-1)\mu\int\limits_{\Omega}(E^{n+1})^{p-2}(\nabla E^{n+1})^{2}d\mathbf{x}\ge0,\qquad(16)$$

Taking into account (16) from (15) we have

$$2\int\limits_{\Omega}(E^{n+1})^{p}d\mathbf{x}+\tau p\mu\int\limits_{\Omega}(E^{n+1})^{p-1}\sum_{\alpha=1}^{3}(\nabla U_{\alpha}^{n+1})^{2}d\mathbf{x}+$$

$$+\tau p(p-1)\mu\int\limits_{\Omega}(E^{n+1})^{p-2}(\nabla E^{n+1})^{2}d\mathbf{x}\le$$

$$\le p\int\limits_{\Omega}(E^{n+1})^{p-1}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}d\mathbf{x}+\tau p\int\limits_{\Omega}(E^{n+1})^{p-1}\mathbf{U}^{n+1}\mathbf{f}^{n}d\mathbf{x}.\qquad(17)$$

The right-hand sides (17) are estimated using the Holder inequality, i.e.

$$p\int\limits_{\Omega}\left(E^{n+1}\right)^{p-1}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1/2}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}.$$

Hence, taking into account the estimate (8) for the fractional step $n+1/2$, we write

$$p\int\limits_{\Omega}\left(E^{n+1}\right)^{p-1}\mathbf{U}^{n+1}\mathbf{U}^{n+1/2}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}} \tag{18}$$

and

$$\tau p\int\limits_{\Omega}\left(E^{n+1}\right)^{p-1}\mathbf{U}^{n+1}\mathbf{f}^{n}d\mathbf{x}\leq\tau\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}. \tag{19}$$

Now using (16), (18), (19) from inequalities (17) we obtain

$$\frac{p}{2^{p-1}}\int\limits_{\Omega}\left(\mathbf{U}^{n+1}\right)^{2p}d\mathbf{x}\leq\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}+$$

$$\tau\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{2p}d\mathbf{x}\right)^{\frac{1}{2p}}.$$

Where, dividing both parts by a positive value $\frac{p}{2^{p-1}}\left(\int\limits_{\Omega}\left|\mathbf{U}^{n+1}\right|^{2p}d\mathbf{x}\right)^{\frac{2p-1}{2p}}$, write down

$$\left(\int\limits_{\Omega}\mid\mathbf{U}^{n+1}\mid^{p}d\mathbf{x}\right)^{\frac{1}{p}}\leq\left(\int\limits_{\Omega}\left|\mathbf{U}^{n}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}+\tau\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}},\ \forall p=2k,\ k\in N.$$

Which, summing over $n=0,1,\cdots,M-1$, we have

$$\left(\int\limits_{\Omega}\mid\mathbf{U}^{n+1}\mid^{p}d\mathbf{x}\right)^{\frac{1}{p}}\leq\left(\int\limits_{\Omega}\left|\mathbf{\Phi}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}+T\max_{n}\left(\int\limits_{\Omega}\left|\mathbf{f}^{n}\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}.$$

Or equivalently the norm of the space $L_p(\Omega)$ estimate (9) in Lemma, Q. E. D.

Corollary. For solutions of the splitting method schemes (4)–(5), the following estimates are valid:

$$\max_{0\leq n\leq M-1}\|\mathbf{U}^{n+1}\|_{L_2(\Omega)}\leq\|\mathbf{\Phi}\|_{L_2(\Omega)}+T\max_{0\leq n<M}\|\mathbf{f}^{n}\|_{L_2(\Omega)}\equiv R_1, \tag{20}$$

$$\max_{0\leq n\leq M-1}\|\mathbf{U}^{n+1}\|_{L_4(\Omega)}\leq\|\mathbf{\Phi}\|_{L_4(\Omega)}+T\max_{0\leq n<M}\|\mathbf{f}^{n}\|_{L_4(\Omega)}\equiv R_2. \tag{21}$$

$$\sum_{n=0}^{M-1}\tau\sum_{\alpha=1}^{3}\left\|\nabla U_{\alpha}^{n+1}\right\|_{L_2(\Omega)}^{2}\leq\frac{1}{\mu}\Big(\left\|\mathbf{\Phi}\right\|_{L_2(\Omega)}^{2}+T/2(1+T)\left\|\mathbf{f}^{n}\right\|_{L_2(\Omega)}^{2}\Big)\equiv R_3, \tag{22}$$

$$\sum_{n=0}^{M-1}\tau\left\|\nabla E^{n+1}\right\|_{L_2(\Omega)}^{2}\leq\frac{1}{2\mu}\Big(\left\|\mathbf{\Phi}\right\|_{L_{4(\Omega)}}+TR_2^{3}\left\|\mathbf{f}^{n}\right\|_{L_4(\Omega)}\Big)\equiv R_4. \tag{23}$$

Proof. The estimates (20), (21) follow from (9) respectively at $p = 2$ and $p = 4$. To prove the estimate (22), write (17) when $p = 1$

$$\int\limits_{\Omega} |\mathbf{U}^{n+1}|^2 d\mathbf{x} + \tau\mu \int\limits_{\Omega} \sum_{\alpha=1}^{3} (\nabla U_{\alpha}^{n+1})^2 d\mathbf{x} \le \int\limits_{\Omega} \mathbf{U}^{n+1}\mathbf{U}^{n+1/2} d\mathbf{x} + \tau \int\limits_{\Omega} \mathbf{U}^{n+1}\mathbf{f}^n d\mathbf{x}. \tag{24}$$

Where to the right parts, using successive inequalities $2ab \le (a^2 + b^2)$, (8) and Cauchy-Bunyakovsky, we get

$$\tau\mu \int\limits_{\Omega} \sum_{\alpha=1}^{3} \left(\nabla U_{\alpha}^{n+1}\right)^2 d\mathbf{x} \le$$

$$\le \frac{1}{2} \int\limits_{\Omega} |\mathbf{U}^n|^2 d\mathbf{x} + \tau \parallel \mathbf{U}^{n+1} \parallel_{L_2(\Omega)} \parallel \mathbf{f}^n \parallel_{L_2(\Omega)} .$$

Here, summing up n, we have

$$\frac{1}{2} \int\limits_{\Omega} |\mathbf{U}^{n+1}|^2 \, \mathbf{dx} + \mu \sum_{n=0}^{M-1} \tau \sum_{\alpha=1}^{3} \int\limits_{\Omega} (\nabla U_{\alpha}^{n+1})^2 d\mathbf{x} \le$$

$$\le \frac{1}{2} \int\limits_{\Omega} |\Phi|^2 \, \mathbf{dx} + \sum_{n=0}^{M-1} \tau \left(\int\limits_{\Omega} \left|\mathbf{U}^{n+1}\right|^2 d\mathbf{x} \right)^{\frac{1}{2}} \left(\int\limits_{\Omega} \left|\mathbf{f}^n\right|^2 d\mathbf{x} \right)^{\frac{1}{2}} .$$

Where by virtue of the nonnegativity of the first integral in the left part and estimates (9), we arrive at an inequality (22), i.e.

$$\mu \sum_{n=0}^{M-1} \tau \sum_{\alpha=1}^{3} \left\|\nabla U_{\alpha}^{n+1}\right\|_{L_2(\Omega)}^2 \le \frac{1}{2} \left\|\mathbf{\Phi}\right\|_{L_2(\Omega)}^2 + \max_n \left\|\mathbf{U}^{n+1}\right\|_{L_2(\Omega)} T \max_n \left\|\mathbf{f}^n\right\|_{L_2(\Omega)}.$$

To prove (23) the estimate (17) we write at $p = 2$, replacing n with m

$$\frac{1}{4} \int\limits_{\Omega} |\mathbf{U}^{m+1}|^4 \, \mathbf{dx} + \mu\tau \int\limits_{\Omega} E^{m+1} \sum_{\alpha=1}^{3} \left(\nabla U_{\alpha}^{m+1}\right)^2 d\mathbf{x} + 2\mu\tau \int\limits_{\Omega} \left(\nabla E^{m+1}\right)^2 d\mathbf{x} \le$$

$$\le \frac{1}{4} \int\limits_{\Omega} | \; \mathbf{U}^m \; |^4 \; d\mathbf{x} + \tau \left(\int\limits_{\Omega} \left|\mathbf{U}^{m+1}\right|^4 d\mathbf{x} \right)^{\frac{3}{4}} \left(\int\limits_{\Omega} \left|\mathbf{f}^m\right|^4 d\mathbf{x} \right)^{\frac{1}{4}} .$$

Which is the sum of m from 0 to n,

$$\frac{1}{4} \sum_{m=0}^{n} \int\limits_{\Omega} |\mathbf{U}^{m+1}|^4 \, \mathbf{dx} + \mu \sum_{m=0}^{n} \tau \int\limits_{\Omega} E^{m+1} \sum_{\alpha=1}^{3} (\nabla U_{\alpha}^{m+1})^2 d\mathbf{x} +$$

$$+2\mu \sum_{m=0}^{n} \tau \int\limits_{\Omega} (\nabla E^{m+1})^2 d\mathbf{x} \le \frac{1}{4} \sum_{m=0}^{n} \int\limits_{\Omega} | \; \mathbf{U}^m \; |^4 \; d\mathbf{x} +$$

$$+ \sum_{m=0}^{n} \tau \Big(\int\limits_{\Omega} \left|\mathbf{U}^{m+1}\right|^4 d\mathbf{x} \Big)^{\frac{3}{4}} \Big(\int\limits_{\Omega} \left|\mathbf{f}^m\right|^4 d\mathbf{x} \Big)^{\frac{1}{4}}.$$

Here, as in the previous case, we find

$$2\mu \sum_{m=0}^{n} \tau \int\limits_{\Omega} (\nabla E^{m+1})^2 d\mathbf{x} \le \left\|\mathbf{\Phi}\right\|_{L_{4(\Omega)}} + T \, R_2^3 \max_{0 \le m \le M} \left\|\mathbf{f}^m\right\|_{L_4(\Omega)}.$$

Where follows (23).

Having excluded from the subsystem (4), (5) a vector function with a fractional exponent $\{n+1/2\}$ of the splitting method, we obtain a system of the whole step

$$\mathbf{U}_{\bar{t}}^m - \mu\Delta\mathbf{U}^{m+1} + (\mathbf{U}^m, \nabla)\mathbf{U}^{m+1} - 2\nabla E^{m+1} = \mathbf{f}^m, \quad \mathbf{U}_{\bar{t}}^m = \big(\mathbf{U}^{m+1} - \mathbf{U}^m\big)/\tau, \tag{25}$$

with initial boundary conditions

$$\mathbf{U}^0(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}), \quad \mathbf{U}^m\big|_{\partial\Omega} = 0, \ m = 0, 1, \cdots, M-1. \tag{26}$$

Theorem. If the input data of the problem (1) satisfy the requirements **i**), **ii**) *and* $\partial\Omega \in C^2$, *then there is a strong generalized solution to the* **U** *problem (1) and have place of evaluation of spaces*

$$\mathbf{U}^m \in \mathbf{W}_{2,0}^{2,1}(\Omega) \cap \mathring{\mathbf{J}}_\infty(\Omega), \forall m \in N,$$

$$\big\|\mathbf{U}_{\bar{t}}^m\big\|_{\mathbf{L}_2(\Omega)}^2 \leq \mu \sum_{\alpha=1}^{3} \big\|\nabla\Phi_\alpha\big\|_{\mathbf{L}_2(\Omega)}^2 + 3T\big\|\mathbf{f}^m\big\|_{\mathbf{L}_2(\Omega)}^2 \equiv R_6, \tag{27}$$

$$\big\|\Delta\mathbf{U}^m\big\|_{\mathbf{L}_2(\Omega)}^2 \leq R_6/\mu^2 \equiv R_7, \tag{28}$$

$$\big\|\nabla U_\alpha^m\big\|_{\mathbf{L}_2(\Omega)}^2 \leq R_6/\mu \equiv R_8, \ \alpha = \overline{1,3}, \tag{29}$$

$$\|\mathbf{U}^m\|_{\mathbf{W}_2^2(\Omega)} \leq R_9\|\Delta\mathbf{U}^m\|_{\mathbf{L}_2(\Omega)}, \quad R_9 - const, \ \forall m \in N. \tag{30}$$

Proof. In order to establish inequalities (27)–(29) from the equation (25) we pass to the identity

$$\int\limits_\Omega \big(\mathbf{U}_{\bar{t}}^m - \mu\Delta\mathbf{U}^{m+1}\big)^2\,\mathbf{dx} = \int\limits_\Omega \big(\mathbf{f}^m - (\mathbf{U}^m, \nabla)\mathbf{U}^{m+1} + 2\nabla E^{m+1}\big)^2\,\mathbf{dx}.$$

We square the integrals and from which we turn to inequality

$$\sum_{m=0}^{n} \tau \int\limits_\Omega \Big(\big(\mathbf{U}_{\bar{t}}^m\big)^2 + \big(\mu\Delta\mathbf{U}^{m+1}\big)^2 - 2\mu\mathbf{U}_{\bar{t}}^m\Delta\mathbf{U}^{m+1}\Big)\mathbf{dx} \leq 3T\max_m\|\mathbf{f}^m\|_{L_2(\Omega)} +$$

$$+3\sum_{m=0}^{n}\tau\int_\Omega|(\mathbf{U}^m,\nabla)\mathbf{U}^{m+1}|^2 d\mathbf{x} + 12\sum_{m=0}^{n}\tau\|\nabla E^{m+1}\|_{L_2(\Omega)}^2. \tag{31}$$

Then pair work on the left side, converting with integration by parts, find the inequality

$$-2\mu\sum_{m=0}^{n}\tau\int_\Omega \mathbf{U}_{\bar{t}}^m\Delta\mathbf{U}^{m+1}d\mathbf{x} \geq \mu\sum_{\alpha=1}^{3}\int_\Omega|\nabla U_\alpha^{n+1}|^2 d\mathbf{x} - \mu\sum_{\alpha=1}^{3}\int\limits_\Omega\big|\nabla\Phi_\alpha\big|^2\mathbf{dx}. \tag{32}$$

In the right part of such force on Young's inequality at $\epsilon = 1$ and $p = 2$:

$$3\sum_{m=0}^{n}\tau\int\limits_\Omega|(\mathbf{U}^m,\nabla)\mathbf{U}^{m+1}|^2 d\mathbf{x} \leq 3\max_m\|\mathbf{U}^m\|_{L_2(\Omega)}^2\sum_{m=0}^{n}\tau\sum_{\alpha=1}^{3}\|\nabla U_\alpha{}^{m+1}\|_{L_2(\Omega)}^2 = 3R_1R_3.$$

As a result, from (31), taking into account (32) and estimates from Lemmas we obtain the inequalities (27)–(29) for strong generalized solutions of the problem (1a)–(1b).

Since the boundary of the domain $\partial\Omega \in C^2$ find the estimate (30), using inequalities from [4], just for any functions $U(x) \in W_2^2(\Omega) \cap W_{2,0}^2(\Omega)$:

$$\|\mathbf{U}^m\|_{\mathbf{W}_2^2(\Omega)} \leq R_9\|\Delta\mathbf{U}^m\|_{\mathbf{L}_2(\Omega)}, \ \forall m \in N, \quad R_9 - const.$$

Theorem is proved.

To show the compactness and existence of solutions, we denote a set of approximate solutions of systems with initial-boundary conditions (4)–(7) via $\{\mathbf{U}^\tau\}$, and the predicted values on the interval $[0,T]-$ through $\widetilde{\mathbf{U}}^\tau$.

Of the estimates (27)–(30) implies the uniform boundedness the norms of interpolating functions $\widetilde{\mathbf{U}}^\tau \in \mathbf{W}^{2,1}_{2,0}(\Omega) \cap \mathring{\mathbf{J}}_\infty(\Omega)$. Consequently, the set $\{\widetilde{\mathbf{U}}^\tau\}$ strongly compact in the space $W^1_2(Q)$. From it you can select convergent subsequence. It will converge strongly in $W^1_2(Q)$ to some elements $\mathbf{U}(t,x) \in W^1_2(Q)$.

The second derivatives and nonlinear terms, respectively, will have its weak limits in $L_2(Q)$.

Remark. In [4] for some difference schemes corresponding to three-dimensional system of non-linear Burgers equations, proved stability in the space $\ell_p, \forall p$.

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Ә.Ш. Ақыш (Ақышев)

Навье-Стокс теңдеуінің ыдырату әдісі және әлді шешімнің табылатындығы

Журналдың өткен санындағы автордың мақаласында үшөлшемді Навье-Стокс теңдеулері (НСТ) шешімдерінің қасиеттерінен қысым мен жылдамдық векторы модулі квадратының арақатынас байланысы табылған. Осы нәтиже негізінде НСТ-ның шешілетіндігі көрсетілген. Зерттеушінің таңдаған кеңістігінде үшөлшемді НСТ-ға қойылған есептің әлсіз шешімінің жалқылығы мен әлді шешімінің ұзақ уақыт бойы табылатындығы дәлелденген. Бұл алынған нәтиже НСТ-ның математикалық сапа теориясын дамытуға әрі қарай да өз үлесін қоса бермек. Ал қазіргі шақтағы өзекті мәселелердің негізгісінің бірі — қойылған есептің шешімін табу үшін математикалық әдістердің ыңғайлы біреуін негіздеу. Мақалада НСТ-ның шешімін табуға қолайлы әдіс ретінде ыдырату әдісі таңдалған. Әдіс негізделіп, әлді шешімді табу алгоритмі ұсынылған.

Кілт сөздер: Навье-Стокс теңдеулері, Навье-Стокс теңдеулеріне ыдырату әдісі, Навье-Стокс теңдеулерінің шешімінің табылатындығы, әлді шешімді табу алгоритмі.

А.Ш. Акыш (Акишев)

Метод расщепления и существование сильного решения уравнений Навье-Стокса

В предыдущем номере данного журнала в статье автора установлено соотношение между давлением и квадратом модуля вектора скорости из свойств решений УНС, на основе чего доказаны единственность слабых и существование сильных решений задачи для трехмерных уравнений Навье-Стокса в целом по времени. Результат является вкладом в качественную математическую теорию уравнений Навье-Стокса. Однако одной из актуальных проблем в теории уравнения Навье-Стокса является выбор математического метода для доказательства теоремы существования. В настоящей работе выбран метод расщепления для решения уравнений Навье-Стокса. Дано обоснование этого метода. Показана компактность последовательности решений, тем самым доказано существование сильных решений задачи для трехмерных уравнений Навье-Стокса в целом по времени.

Ключевые слова: уравнения Навье-Стокса, метод расщепления для уравнений Навье-Стокса, компактность, существование сильных решений, алгоритм определения сильных решений.