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GEODESICS IN A TWO-DIMENSIONAL MODEL OF CLOSED SPACETIME

V. V. Arkhipov

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A simplified model of closed spacetime is considered which treats it as being imbedded in a flat, enveloping space of greater dimensionality. It is shown that in the case of expansion of space with acceleration only timelike geodesics have a solution that is infinite in time. Spacelike geodesics in this model are finite in time. This circumstance, in a natural way, forbids motion of free test particles with velocities greater than the speed of light. In principle, the conclusions of this work can be generalized to the case of a space of greater dimensionality.

Keywords: metric, geodesics, gravitation.

Recent achievements in experimental cosmology have made it possible to draw revolutionary conclusions about the need for a deep reassessment of our theoretical ideas about the physics of the Universe. The first steps in this direction were indicated by the discovery of a discrepancy between the distribution of the visible mass of spiral galaxies and the corresponding dynamics based on Newton's laws or the general theory of relativity. The solution to this problem required the introduction of dark matter [1] – matter which does not radiate, but is responsible for a contribution to the gravitational potential. Estimates of the required quantity of dark matter led to the conclusion that the geometry of the Universe should have a very weak curvature, closely approaching to an Euclidean one. However, estimates of the rate of expansion of the Universe based on an analysis of the brightness of supernovae led to the conclusion of accelerated expansion, which contradicts the idea that only gravitational forces of attraction act on cosmic scales [2]. There is no unified approach to explain these results [3]. Among the main approaches being considered, we may single out a return to the equations of the general theory of relativity with a Λ-term, where, it will be recalled, this Λ-term is responsible for attractive forces growing with distance, and also various combinations of geometric approaches introducing exotic types of matter, such as mirror matter [4], etc. Great weight is also attached to alternatives to general relativity theory, such as MOND (modified Newtonian dynamics) [5].

At present, the geometry of the Universe remains an open question. The observed flatness of the visible part of the Universe has stimulated the proposal of models that allow this property to be combined with the closure of the spatial component of spacetime (see, for example, [6]). The idea of the closure of space has no serious physical basis with the exception of the difficulty of perceiving an infinite Universe and a number of paradoxes which all have completely acceptable explanations.

In this work we consider a simplified model of a completely symmetric, one-dimensional, spatially closed Universe (toy model), which enables us to investigate the question of the general behavior of geodesics in the case of accelerated expansion.

We consider a graphic model of spacetime with closed one-dimensional space. We assume that this space is imbedded in flat spacetime with the metric

Academician E. A. Buketov Karaganda State University, Karaganda, Kazakhstan, e-mail: midav_73@mail.ru.
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$$ds^2 = dt^2 - dx^2 - dy^2 .$$

The circle $x^2 + y^2 = R^2$ will play the role of a closed subspace with maximum symmetry. Transforming to a polar coordinate system represented by the one variable φ, and bearing in mind the possible dependence of R on time, $R(t)$, we have

$$ds^2 = \left(1 - \left(\frac{\partial R}{\partial t}\right)^2\right) dt^2 - R^2 d\varphi^2 .$$

Thus, the metric has the form

$$g_{\mu\nu} = \mathrm{diag}\left(1 - \left(\frac{\partial R}{\partial t}\right)^2, -R^2\right). \qquad (1)$$

For the connection we have the following nonzero components:

$$\Gamma^0_{00} = -\frac{\partial R / \partial t}{1 - (\partial R / \partial t)^2} \frac{\partial^2 R}{\partial t^2}, \ \Gamma^0_{11} = \frac{\partial R / \partial t}{1 - (\partial R / \partial t)^2} R, \ \Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{R} \frac{\partial R}{\partial t} .$$

From the equations for the geodesics

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0$$

we obtain a system of the form

$$\frac{d^2 t}{ds^2} - \frac{\partial R / \partial t}{1 - (\partial R / \partial t)^2} \frac{\partial^2 R}{\partial t^2} \left(\frac{dt}{ds}\right)^2 + \frac{\partial R / \partial t}{1 - (\partial R / \partial t)^2} R \left(\frac{d\varphi}{ds}\right)^2 = 0,$$
$$\frac{d^2 \varphi}{ds^2} + \frac{2}{R} \frac{\partial R}{\partial t} \frac{dt}{ds} \frac{d\varphi}{ds} = 0. \qquad (2)$$

The solution will depend on the concrete form of $R(t)$. Note that the geodesics of the enveloping space have the form of straight lines. It is possible to preserve this property for isotropic geodesics, bearing in mind their *selectness* for pseudo-Euclidean spaces. We choose the dependence $R(t)$ such that the closed space is formed by isotropic $\left(t = \sqrt{x^2 + y^2}\right)$ geodesics as depicted in Fig. 1. For convenience, we choose a geodesic lying parallel to the tOy plane and intersecting the Ox axis at the point $R_0 = R(0)$.

Noting that $|y| = t$, we obtain

$$R(t) = \sqrt{R_0^2 + t^2} . \qquad (3)$$

Thus, from Eqs. (2) we obtain the following system for all geodesics:

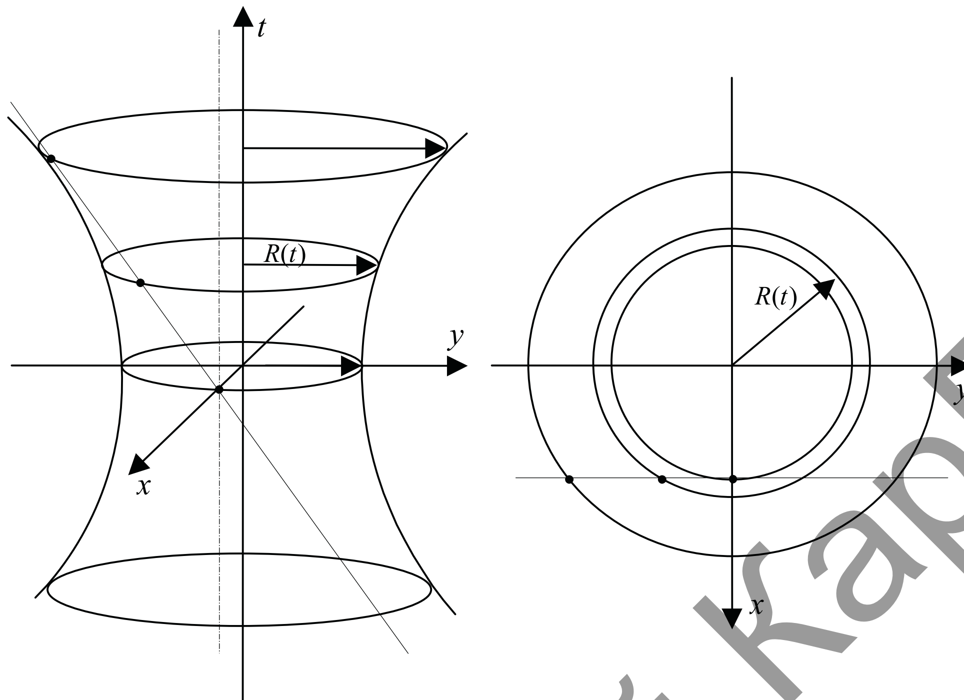

Fig. 1. Schematic view of closed spacetime as the figure of rotation with respect to the time axis. A curved surface is formed by isotropic geodesics of the enveloping flat spacetime.

$$\frac{d^2t}{ds^2}-\frac{t}{R_0^2+t^2}\left(\frac{dt}{ds}\right)^2+\frac{R_0^2+t^2}{R_0^2}\left(\frac{d\varphi}{ds}\right)^2=0,$$
$$\frac{d^2\varphi}{ds^2}+\frac{2t}{R_0^2+t^2}\frac{dt}{ds}\frac{d\varphi}{ds}=0. \tag{4}$$

The latter of these two equations is easily transformed as follows:

$$\left(R_0^2+t^2\right)\frac{d^2\varphi}{ds^2}+\frac{d(R_0^2+t^2)}{ds}\frac{d\varphi}{ds}=\frac{d}{ds}\left(\left(R_0^2+t^2\right)\frac{d\varphi}{ds}\right)=0\ .$$

Consequently, we obtain

$$\frac{d\varphi}{ds}=\frac{R_0C}{\left(R_0^2+t^2\right)}, \tag{5}$$

where $C=\text{const}$. Thus, the first equation of system (4) can be rewritten in the form

$$\frac{d^2t}{ds^2}-\frac{t}{R_0^2+t^2}\left(\frac{dt}{ds}\right)^2+\frac{C^2}{R_0^2+t^2}=0\ .$$

It is not hard to show that this equation can be represented as

$$\frac{d^2}{ds^2}\left(\operatorname{arcsinh}\frac{t}{R_0}\right)+\frac{tC^2}{\left(R_0^2+t^2\right)^{3/2}}=0 .$$

By means of the substitution of variables $y=\operatorname{arcsinh}\left(t/R_0\right)$ the last equation reduces to an autonomous differential equation:

$$y''=-\frac{\sinh y}{\cosh^3 y}\left(\frac{C}{R_0}\right)^2 .$$

Its solution has the form

$$\int\left[C_1+2\int\left[-\frac{\sinh y}{\cosh^3 y}\left(\frac{C}{R_0}\right)^2\right]dy\right]^{-1/2}dy=C_2\pm s ,$$

where $C_{1,2}=\mathrm{const}$. The integral inside the outer brackets can be easily evaluated:

$$\int\left[C_1+\frac{C^2}{\cosh^2 y}\right]^{-1/2}dy=C_2\pm s .$$

After some transformations, we obtain

$$\sinh y=\sqrt{1+\frac{C^2}{C_1R_0^2}}\sinh\left(\sqrt{C_1}(C_2\pm s)\right);$$

hence,

$$t=\sqrt{R_0^2+\frac{C^2}{C_1}}\sinh\left(\sqrt{C_1}(C_2\pm s)\right), \qquad (6)$$

where the “+” and “–” signs, obviously, correspond to motion in the positive direction of time and opposite to it. In light of this treatment, in what follows we will keep only the “+” sign.

The obtained result can be verified by constructing a two-dimensional analog of the 4-velocity:

$$u^\alpha=\left(\frac{dt}{ds},\frac{d\varphi}{ds}\right).$$

For the modulus $|u|^2=g_{\alpha\beta}u^\alpha u^\beta$, employing Eqs. (5) and (6) and the components of metric (1) with the chosen dependence (3) taken into account

$$g_{\mu\nu}=\operatorname{diag}\left(1-\frac{t^2}{R_0^2+t^2},-(R_0^2+t^2)\right),$$

we obtain

$$|u|^2 = R_0^2 C_1 .$$

Hence it is clear that the constant C_1 plays the role of the scale of the parameter of the geodesic s. In what follows we take s as a natural parameter, which implies that $|u|^2 = 1$ and, consequently, $C_1 = 1/R_0^2$.

Substituting expression (6) into Eq. (5), we obtain the following expression for the coordinate φ:

$$\varphi = \frac{1}{R_0}\int \frac{C ds}{1+(1+C^2)\sinh^2((C_2+s)/R_0)},$$

whose solution has the form

$$\varphi = \arctan\left(C \tanh\left(\frac{s+C_2}{R_0} \right)\right) + C_3 .$$

Let us consider different geodesics, imposing initial conditions at the time $t = 0$. Without loss of generality, we can set $C_2 = 0$, which corresponds to a choice of the reference point of the parameter s, and $C_3 = 0$, which corresponds to a choice of the geodesic passing through the *Ox* axis. The parametric equations of such a geodesic take the form

$$t = R_0\sqrt{1+C^2}\sinh(s/R_0), \ \varphi = \arctan\left(C \tanh\left(s/R_0\right)\right).$$

Hence we obtain the equation of the trajectory

$$\varphi(t) = \arctan\left(C \frac{t}{\sqrt{R_0^2(1+C^2)+t^2}} \right). \tag{7}$$

In order to determine the constant C, we introduce the initial velocity of the trajectory $\omega_0 = \left.\frac{d\varphi}{dt}\right|_{t=0}$:

$$\omega_0 = \frac{C}{R_0\sqrt{1+C^2}}, \ C = \frac{R_0\omega_0}{\sqrt{1-R_0^2\omega_0^2}} .$$

Taking into account the zero rate of expansion of space at the initial time $t = 0$, it is possible to interpret $R_0\omega_0$ as the ordinary velocity of the test particle v_0 in the accompanying coordinate system. It is necessary to take $v_0 < 1$ in order for the geodesic to be timelike. Then from Eq. (7) we obtain

$$\varphi(t) = \arctan\left(\frac{v_0 t}{\sqrt{R_0^2 + (1-v_0^2)t^2}} \right). \tag{8}$$

For an isotropic geodesic, $v_0 = 1$, the equation of the geodesic (8) can be written in the following simple form:

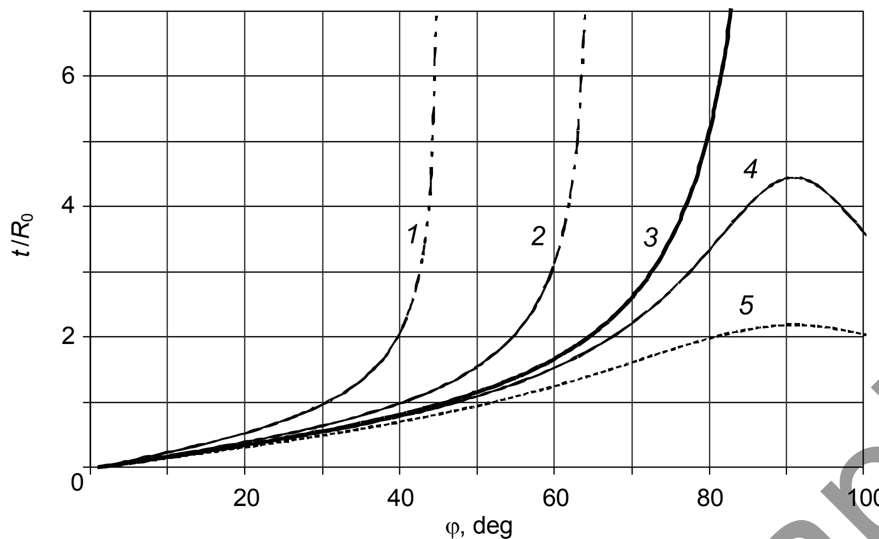

Fig. 2. Graph of geodesics for values of the initial velocity v_0 : curves *1* and *2* are for $v_0 = 0.7$ and 0.9 (timelike), curve *3* is for $v_0 = 1$ (isotropic), and curves *4* and *5* are for $v_0 = 1.025$ and 1.1 (spacelike). Values of t / R_0 are plotted along the ordinate, and the coordinate φ is plotted along the abscissa in degrees. The spacelike curves have an extremum at $\varphi = \pi / 2$.

$$\tan \varphi = \frac{t}{R_0} .$$

That means that with the passage of time isotropic geodesics approach the *lateral line* of spacetime $\varphi = \pi / 2$, without glancing beyond it.

If the initial velocity of the geodesic is greater than unity, i.e., it is spacelike, then we obtain motion that is finite in time and described by the following expression obtained from Eq. (8):

$$t = R_0 \frac{\sin \varphi}{\sqrt{v_0^2 - \sin^2 \varphi}} .$$

It can be easily found that the limiting value of the time is

$$t_{ex} = R_0 / \sqrt{v_0^2 - 1} . \tag{9}$$

That means, the larger the value of the superluminal velocity v_0 , the smaller is t_{ex} . Figure 2 presents a graph of geodesics for a few values of v_0 .

Thus, if it is granted that the model presented here can be generalized to the case of a large number of spatial dimensions, then we can draw the following conclusion: In a Universe expanding with positive acceleration, all particles having initial velocity greater than that of light (tachyons) should be annihilated over the course of time, that is, change their direction of motion to be opposite the direction of time, if we resort to the Feynman interpretation. The lifetime of such particles will depend strongly on their initial velocity. According to Eq. (9), the larger their velocity, the faster they disappear.

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