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TRIGONOMETRIC WIDTHS OF THE NIKOL'SKII–BESOV CLASSES IN THE LEBESGUE SPACE WITH MIXED NORM

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We establish exact-order estimates for the trigonometric widths of the Nikol'skii–Besov classes of periodic functions of many variables in the Lebesgue space with mixed norm.

Introduction

Let $\overline{x} = (x_1, \ldots, x_m) \in \mathbb{T}^m = [0, 2\pi)^m$. By $L_{\bar{p}}(\mathbb{T}^m)$, we denote a space of Lebesgue-measurable functions $f(\bar{x})$ defined on $\mathbb{R}^m$, 2π-periodic in each variable, and such that

$$\|f\|_{\bar{p}} = \left[\int_0^{2\pi} \left[\ldots \left[\int_0^{2\pi} \left| f(\bar{x}) \right|^{p_1} dx_1 \right]^{p_2/p_1} \ldots \right]^{p_m/(p_{m-1})} dx_m \right]^{1/p_m} < \infty,$$

where $\overline{p} = (p_1, \ldots, p_m)$, $1 \leq p_j < \infty$, $j = 1, \ldots, m$ (see [1, p. 128]). In the case where $p_1 = p, \ldots, p_m = p$, instead of $\| \cdot \|_{\bar{p}}$, $L_{\bar{p}}(\mathbb{T}^m)$ and $B^r_{\bar{p},\theta}$, we use the notation $\| \cdot \|_p$, $L_p(\mathbb{T}^m)$ and $B^r_{p,\theta}$.

The function $f \in L_1(\mathbb{T}^m) = L(\mathbb{T}^m)$ is expanded in the Fourier series

$$\sum_{\overline{n} \in \mathbb{Z}^m} a_{\overline{n}}(f) e^{i\langle \overline{n}, \overline{x} \rangle},$$

where $\langle \bar{n}, \bar{x} \rangle = \sum_{j=1}^m n_j x_j$, $a_{\overline{n}}(f)$ are the Fourier coefficients of the function $f \in L_1(\mathbb{T}^m)$ in the multiple trigonometric system $\left\{ e^{i\langle \overline{n}, \overline{x} \rangle} \right\}_{\overline{n} \in \mathbb{Z}^m}$, and $\mathbb{Z}^m$ is a space of points from $\mathbb{R}^m$ with integer-valued coordinates.

For the function $f \in L(\mathbb{T}^m)$ and number $s \in \mathbb{Z}_+$, we set

$$\sigma_s(f, \overline{x}) = \sum_{\overline{n} \in \rho(s)} a_{\overline{n}}(f) e^{i\langle \overline{n}, \overline{x} \rangle},$$

$$\rho(s) = \left\{ \overline{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m : \left[2^{s-1} \right] \leq \max_{j=1,\ldots,m} |k_j| < 2^s \right\},$$

where $[a]$ is the integer part of the number a.

We define the Nikol'skii–Besov classes [1, 2] as follows: Let $1 < p_j < \infty$, $j = 1, \ldots, m$, $1 \leq \theta \leq \infty$, and $r > 0$. Then,

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$$B^r_{\bar{p},\theta} = \left\{ f \in L_{\bar{p}}(\mathbb{T}^m) : \left(\sum_{s\in\mathbb{Z}_+} 2^{sr\theta} \left\| \sigma_s(f) \right\|^\theta_{\bar{p}} \right)^{1/\theta} \leq 1 \right\},$$

$$H^r_{\bar{p}} = \left\{ f \in L_{\bar{p}}(\mathbb{T}^m) : \sup_{s\in\mathbb{Z}_+} 2^{sr} \left\| \sigma_s(f) \right\|_{\bar{p}} \leq 1 \right\}.$$

It is known that, for $1 < \theta_1 < \theta_2 < \infty$, the following imbeddings are true:

$$B^r_{\bar{p},1} \subset B^r_{\bar{p},\theta_1} \subset B^r_{\bar{p},\theta_2} \subset B^r_{\bar{p},\infty} = H^r_{\bar{p}}. \tag{1}$$

Consider a given class $F \subset L_{\bar{p}}(\mathbb{T}^m)$. A trigonometric n-width $d^T_n(F, L_{\bar{p}})$ of the class F in the space $L_{\bar{p}}(\mathbb{T}^m)$ is defined as

$$d^T_n(F, L_{\bar{p}}) = \inf_{\Omega_n} \sup_{f\in F} \inf_{t(\Omega_n)} \left\| f - t(\Omega_n) \right\|_{\bar{p}}, \tag{2}$$

where

$$t(\Omega_n, \bar{x}) = \sum_{j=1}^{n} c_j e^{i\langle \bar{k}^{(j)}, \bar{x}\rangle},$$

$\Omega_n = \left\{ \bar{k}^{(1)}, \bar{k}^{(2)}, \ldots, \bar{k}^{(n)} \right\}$ is a family of vectors $\bar{k}^{(j)} = \left(k^{(j)}_1, \ldots, k^{(j)}_m \right)$, $j = 1, \ldots, n$, with integer-valued coordinates, and c_j are arbitrary numbers.

For the first time, the notion of a trigonometric width in the one-dimensional case was introduced by Nikol'skii in [3] who established estimates for some classes in the space of continuous functions. For functions of many variables, the trigonometric widths of the Sobolev classes $W^{\bar{r}}_p$ and the Nikol'skii classes $H^{\bar{r}}_p$ were investigated by Bugrov [4], Belinskii [5, 6], Maiorov [7], Magaril-Il'yaev [8], and Temlyakov [9]. For the Besov classes $B^{\bar{r}}_{p,\theta}$, these widths were studied by Romanyuk [10]. For the generalized Nikol'skii–Besov classes, the trigonometric widths were investigated by Stasyuk [11, 12] and Bazarkhanov [13].

For the class $B^r_{p,\theta}$, where r, p, and θ are certain numerical parameters, the following theorem is proved in [14]:

Theorem [14]. *Let* $1 \leq p < 2 \leq q < \dfrac{p}{p-1}$, $1 \leq \theta \leq \infty$, *and* $r > m$. *Then*

$$d^T_n(B^r_{p,\theta}, L_q) \asymp n^{-\frac{r}{m} + \frac{1}{p} - \frac{1}{2}}.$$

The aim of the present paper is to determine the exact order of the trigonometric width of the class $B^r_{\bar{p},\theta}$ in the space $L_{\bar{q}}(\mathbb{T}^m)$.

In the case where inequality $B \geq C_1 A$ (or $B \leq C_2 A$) is true, we often write $B \gg A$ (or $B \ll A$, respectively). The notation $A \asymp B$ means that $A \ll B$ and $B \ll A$.

Auxiliary Statements

Let $f \in L_{\bar{p}}(\mathbb{T}^m)$ and let $\left\{ \bar{k}^{(j)} \right\}^M_{j=1}$ be a system of vectors $\bar{k}^{(j)} = \left(k^{(j)}_1, \ldots, k^{(j)}_m \right)$ with integer-valued coordinates. Consider a quantity

$$e_M(f)_{\bar{p}} = \inf_{\bar{k}^{(j)}, b_j} \left\| f - \sum_{j=1}^{M} b_j e^{i\langle \bar{k}^{(j)}, \bar{x} \rangle} \right\|_{\bar{p}},$$

where b_j are arbitrary numbers. The quantity $e_M(f)_{\bar{p}}$ is called the best M-term trigonometric approximation of the function $f \in L_{\bar{p}}(\mathbb{T}^m)$. For a given class $F \subset L_{\bar{p}}(\mathbb{T}^m)$, we set

$$e_M(F)_{\bar{p}} = \sup_{f \in F} e_M(f)_{\bar{p}}. \tag{3}$$

According to definitions (2) and (3), the quantities $d_M^T(F, L_{\bar{p}})$ and $e_M(F)_{\bar{p}}$ are related by the inequality

$$e_M(F)_{\bar{p}} \le d_M^T(F)_{\bar{p}}. \tag{4}$$

To get a lower bound for the trigonometric width of the class $B^r_{\bar{p},\theta}$, we need the following assertion:

Theorem 1 [15]. *Let* $\bar{p} = (p_1, \ldots, p_m)$, $\bar{q} = (q_1, \ldots, q_m)$, $1 < p_j \le 2 < q_j < \infty$, $j = 1, \ldots, m$, $1 \le \theta \le \infty$. *If*

$$r > \sum_{j=1}^{m} \frac{1}{p_j},$$

then

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \asymp M^{-\frac{1}{m}\left(r - \sum_{j=1}^{m}\left(\frac{1}{p_j} - \frac{1}{2}\right)\right)}.$$

Proof. We restrict ourselves to finding the lower bound frequently is used in what follows. To do this, we need the relation (see [16, p. 25])

$$e_M(f)_{\bar{q}} = \inf_{\Omega_M} \sup_{P \in L^{\perp}, \|P\|_{\bar{q}'} \le 1} \left| \int_{\mathbb{T}^m} f(\bar{x}) \bar{P}(\bar{x}) d\bar{x} \right|, \tag{5}$$

where $\bar{q}' = (q_1', \ldots, q_m')$, $\dfrac{1}{q_j} + \dfrac{1}{q_j'} = 1$, $j = 1, \ldots, m$, $L^{\perp}$ is the set of functions orthogonal to a subspace of trigonometric polynomials with numbers of harmonics from the set Ω_M.

Since the estimate for the quantity $e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}}$ is independent of θ, by virtue of (1), it suffices to establish the lower bound for $B^r_{\bar{p},1}$.

For a natural number M, we choose a number $n \in \mathbb{N}$ such that $M \asymp 2^{nm}$ and $2M \le \sharp\rho(n)$, where $\sharp\rho(n)$ is the number of elements of the set $\rho(n)$.

Consider a function

$$f_1(\bar{x}) = 2^{-n\left(r + \sum_{j=1}^{m}\left(1 - \frac{1}{p_j}\right)\right)} \sum_{\bar{k} \in \rho(n)} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Then $\left\|\sigma_s(f_1)\right\|_{\bar{p}} = 0$ if $s \ne n$ and

$$\left\|\sigma_n(f_1)\right\|_{\bar{p}} = 2^{-n\left(r + \sum_{j=1}^{m}\left(1 - \frac{1}{p_j}\right)\right)} \prod_{j=1}^{m} \left\| \sum_{k_j = 2^{n-1}}^{2^n - 1} e^{i k_j x_j} \right\|_{p_j}.$$

By virtue of the estimate for the norm of the Dirichlet kernel (see [17, p. 181]), we obtain

$$\left\|\sum_{k_j=2^{n-1}}^{2^n-1} e^{ik_jx_j}\right\|_{p_j} \ll 2^{n\left(1-\frac{1}{p_j}\right)}$$

for $p_j \in (1,\infty),\ j=1,\dots,m$. Hence,

$$\left\|\sigma_n(f_1)\right\|_{\bar{p}} \ll 2^{-nr}$$

and, therefore,

$$\sum_{s=1}^{\infty} 2^{sr}\left\|\sigma_s(f_1)\right\|_{\bar{p}} \le C_1,$$

i.e., the function $C_1^{-1} f_1 \in B^r_{\bar{p},1}$.

Further, we consider two functions

$$v_1(\bar{x}) = \sum_{\bar{k}\in\rho(n)} e^{i\langle\bar{k},\bar{x}\rangle},$$

$$u_1(\bar{x}) = \sum_{\bar{k}\in\rho(n)\cap\Omega_M} e^{i\langle\bar{k},\bar{x}\rangle}$$

and set $w_1(\bar{x}) = v_1(\bar{x}) - u_1(\bar{x})$. By virtue of the Parseval equality, we obtain

$$\|u_1\|_2 = (\pi)^m \left(\sum_{\bar{k}\in\rho(n)\cap\Omega_M} 1\right)^{1/2} \ll M^{1/2},$$

$$\|v_1\|_2 = (\pi)^m \left(\sum_{\bar{k}\in\rho(n)} 1\right)^{1/2} \ll 2^{nm/2}.$$

According to the property of the norm, these relations yield the following inequality:

$$\|w_1\|_2 \le \|v_1\|_2 + \|u_1\|_2 \le C_2 2^{nm/2}.$$

Hence, the function $P_1(\bar{x}) = C_2^{-1} 2^{-nm/2} w_1(\bar{x})$ satisfies (5) for $q_j = 2,\ j=1,\dots,m$. Since $2 < q_j,\ j = 1,\dots,m$, we have $e_M\left(f_1\right)_2 \ll e_M\left(f_1\right)_{\bar{q}}$. By using relation (5), we get

$$e_M(f_1)_{\bar{q}} \gg e_M(f_1)_2 \gg \inf_{\Omega_M} \int_{\mathbb{T}^m} f_1(\bar{x})\bar{P}_1(\bar{x})d\bar{x}$$

$$= C_2^{-1} 2^{-nm/2} 2^{-n\left(r+\sum_{j=1}^m \left(1-\frac{1}{p_j}\right)\right)} \inf_{\Omega_M}\left[\sharp\rho(n) - \sharp(\rho(n)\cap\Omega_M)\right]$$

$$\gg 2^{-nm/2}2^{-n\left(r+\sum_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)}\left[\sharp\rho(n)-M\right]$$

$$\gg 2^{-nm/2}2^{-n\left(r+\sum_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)}\left[\sharp\rho(n)-\frac{\sharp\rho(n)}{2}\right]\gg 2^{-nm/2}2^{-n\left(r-\sum_{j=1}^{m}\frac{1}{p_j}\right)}.$$

In view of relations (1), the last inequalities, and the fact that $2^{nm}\asymp M,$ we find

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q}\geq e_M\left(B^r_{\bar p,1}\right)_{\bar q}\gg e_M\left(f_1\right)_{\bar q}\gg M^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)}.$$

Theorem 1 is proved.

Remark 1. In the case where $p_1=\ldots=p_m=p$ and $q_1=\ldots=q_m=q,$ Theorem 1 is proved in [18]. The estimates for the quantity $e_M\left(B^r_{p,\theta}\right)_q$ in the case

$$\frac{m}{p}-\frac{m}{q}<r<\frac{m}{p}$$

are established in [19]. In the one-dimensional case, Theorem 1 is proved in [6]. Moreover, Theorem 1 is presented in [15] for the other relations between the parameters p_j and $q_j,\ j=1,\ldots,m.$

Theorem B [20]. *Let* $\bar n=(n_1,\ldots,n_m),\ n_j\in\mathbb{N},\ j=1,\ldots,m,$ *and let*

$$T_{\bar n}(\bar x)=\sum_{|k_j|\leq n_j,j=1,\ldots,m}c_{\bar k}e^{i\langle\bar k,\bar x\rangle}.$$

Then, for $1\leq p_j<q_j\leq\infty,\ j=1,\ldots,m,$ *the inequality*

$$\|T_{\bar n}\|_{\bar q}\leq 2^m\prod_{j=1}^{m}n_j^{1/p_j-1/q_j}\|T_{\bar n}\|_{\bar p}$$

is true.

Let Ω_M be a set that contains at most M vectors $\bar k=(k_1,\ldots,k_m)$ with integer-valued coordinates. The following lemma is true:

Lemma A [21]. *Let* $2\leq q<\infty$. *Then, for any trigonometric polynomial*

$$P(\Omega_M,\bar x)=\sum_{j=1}^{M}e^{i\langle\bar k^{(j)},\bar x\rangle}$$

and any natural number $N\leq M,$ *there exists a trigonometric polynomial* $P(\Omega_N,\bar x)$ *containing at most* N *harmonics such that*

$$\left\|P(\Omega_M)-P(\Omega_N)\right\|_q\ll MN^{-1/2}.$$

Furthermore, $\Omega_N\subset\Omega_M,$ *all coefficients* $P(\Omega_N,\bar x)$ *are equal, and their absolute value does not exceed* $MN^{-1}.$

Main Results

First, we prove an auxiliary statement necessary in what follows in the proof of the main results. Consider a trigonometric polynomial

$$t_s(\bar{x}) = \sum_{\bar{k}\in\rho(s)} e^{i\langle\bar{k},\bar{x}\rangle}.$$

Let $t(\Omega_{n_s};\bar{x})$ be a trigonometric polynomial approximating the polynomial $t_s(\bar{x})$ according to Lemma A, i.e.,

$$\big\|t_s - t(\Omega_{n_s})\big\|_{\bar{q}} \le \big\|t_s - t(\Omega_{n_s})\big\|_{\tilde{q}} \ll 2^{sm} n_s^{-1/2}, \tag{6}$$

where $\tilde{q} = \max\{q_j \colon j = 1,\dots,m\}$, $\Omega_{n_s} \subset \rho(s)$, all coefficients of the polynomial $t(\Omega_{n_s};\bar{x})$ are equal, and their absolute value does not exceed $2^{sm}n_s^{-1}$.

Consider an operator T_s of the form

$$T_s f(\bar{x}) = f(\bar{x}) * \big(t_s(\bar{x}) - t(\Omega_{n_s},\bar{x})\big),$$

where the symbol $*$ denotes the operation of convolution of two functions, i.e.,

$$(\varphi * g)(\bar{x}) := (2\pi)^{-m}\int\limits_{\mathbb{T}^m} \varphi(\bar{y})g(\bar{x}-\bar{y})d\bar{y}$$

for $\varphi, g \in L(\mathbb{T}^m)$.

The following statement is true:

Lemma 1. *Let $1 < p_j < 2 < q_j < \dfrac{p_j}{p_j-1} = p'_j$, $j = 1,\dots,m$. Then the norm of the operator T_s acting from $L_{\bar{p}}(\mathbb{T}^m)$ into $L_{\bar{q}}(\mathbb{T}^m)$ satisfies the inequality*

$$\|T_s\|_{\bar{p}\to\bar{q}} = \sup_{\|f\|_{\bar{p}}\le 1} \|T_s f\|_{\bar{q}} \ll 2^{sm} n_s^{-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)}.$$

Proof. According to an analog of the Riesz–Thorin theorem in the Lebesgue space with mixed norm (see [22]), we can write

$$\|T_s\|_{\bar{p}\to\bar{q}} \le \|T_s\|_{2\to 2}^{1-\lambda}\|T_s\|_{1\to\bar{q}^*}^{\lambda}, \tag{7}$$

where $0 < \lambda < 1$ and the coordinates $\bar{q}^* = (q_1^*,\dots,q_m^*)$ satisfy the equality

$$\frac{1}{q_j} = \frac{1-\lambda}{2} + \frac{\lambda}{q_j^*}, \quad j = 1,\dots,m.$$

We choose

$$\lambda = \frac{2}{m}\sum_{j=1}^{m}\left(\frac{1}{p_j} - \frac{1}{2}\right).$$

The coefficients of the polynomial $t_s(\bar{x}) - t(\Omega_{n_s}, \bar{x})$ are equal and their absolute values do not exceed $2^{(s+1)m} n_s^{-1} + 1$. Hence, by virtue of the Parseval equality, we obtain

$$\|T_s\|_{2\to 2} \ll 2^{sm} n_s^{-1}. \tag{8}$$

Further, by using the generalized Minkowski inequality (see [1, p. 27]) and (6), we get

$$\|T_s f\|_{\bar{q}^*} \le \|f\|_1 \big\|t_s - t(\Omega_{n_s})\big\|_{\bar{q}^*} \ll \|f\|_1 2^{sm} n_s^{-1/2}.$$

Therefore,

$$\|T_s\|_{1\to \bar{q}^*} \ll 2^{sm} n_s^{-1/2}. \tag{9}$$

Substituting (8) and (9) in (7), we obtain

$$\|T_s\|_{\bar{p}\to\bar{q}} \ll \left(2^{sm} n_s^{-1}\right)^{1-\lambda} \left(2^{sm} n_s^{-1/2}\right)^{\lambda} = C 2^{sm} n_s^{\frac{\lambda}{2}-1} = C 2^{sm} n_s^{-\frac{1}{m}\sum_{j=1}^m \left(\frac{1}{2}+\frac{1}{p_j'}\right)}.$$

Lemma 1 is proved.

Remark 2. In the case where $p_1 = \ldots = p_m = p$ and $q_1 = \ldots = q_m = q$, Lemma 1 was proved in [14].

We now formulate and prove the main result of the present paper.

Theorem 2. *Let* $1 < p_j < 2 \le q_j < \dfrac{p_j}{p_j - 1}$, $j = 1, \ldots, m$, $1 \le \theta \le \infty$, *and* $r > m$. *Then*

$$d_n^T(B^r_{\bar{p},\theta}, L_{\bar{q}}) \asymp n^{-\frac{r}{m}+\frac{1}{m}\sum_{j=1}^m \left(\frac{1}{p_j}-\frac{1}{2}\right)}.$$

Proof. In view of (4), the lower bound was, in fact, established in deducing the corresponding lower bound in Theorem 1.

We establish the upper bound. For a number $n \in \mathbb{N}$, we choose a natural number l such that $2^{(l-1)m-1} \le n < 2^{lm-1}$. We set

$$\alpha = \frac{\dfrac{r}{m} - \dfrac{1}{m}\sum_{j=1}^m \left(\dfrac{1}{p_j} - \dfrac{1}{2}\right)}{\dfrac{r}{m} - \dfrac{1}{m}\sum_{j=1}^m \left(\dfrac{1}{p_j} - \dfrac{1}{q_j}\right)}.$$

For $s = 0, 1, 2, \ldots,$ we denote $n_s = 2^{sm}$ if $0 \le s < l$,

$$n_s = \left[2^{lr} 2^{sm\left(1-\frac{r}{m}\right)}\right]$$

if $l \le s \le [\alpha l] + 1$, and $n_s = 0$ if $s > [\alpha l] + 1$, where $[y]$ is the integer part of the number y.

Since $r > m,$ we get

$$\sum_s n_s \le \sum_{s=0}^{l-1} 2^{sm} + \sum_{s=l}^{[\alpha l]+1} 2^{lr} 2^{sm\left(1-\frac{r}{m}\right)} \ll C\left\{2^{lm} + 2^{lr}2^{lm\left(1-\frac{r}{m}\right)}\right\} \le 2C2^{lm} \asymp n. \tag{10}$$

Consider the sets

$$P = \bigcup_{0\le s<l} \rho(s) \qquad \text{and} \qquad Q = \bigcup_{l\le s\le[\alpha l]+1} \Omega_{n_s}.$$

We construct a subspace of trigonometric polynomials with harmonics from the set $P \cup Q$ such that the approximation of the class $H_{\bar{p}}^r$ in the space $L_{\bar{q}}(\mathbb{T}^m)$ by this subspace realizes the order of the quantity $d_n^T(H_{\bar{p}}^r, L_{\bar{q}}).$

Let $f \in H_{\bar{p}}^r.$ Consider the approximation of the function f by polynomials of the form

$$t(\bar{x}) = \sum_{s=0}^{l-1} \sigma_s(f,\bar{x}) + \sum_{s=l}^{[\alpha l]+1} \big(t(\Omega_{n_s};\bar{x}) * \sigma_s(f,\bar{x})\big).$$

By virtue of relation (10), the number of harmonics in the polynomial $t(\bar{x})$ does not exceed $n.$ Thus, according to the property of the norm, we find

$$\|f-t\|_{\bar{q}} \le \left\|\sum_{s=l}^{[\alpha l]+1}\left(\sigma_s(f,\bar{x}) - \Big(t(\Omega_{n_s};\bar{x}) * \sigma_s(f,\bar{x})\Big)\right)\right\|_{\bar{q}} + \left\|\sum_{s>[\alpha l]+1}\sigma_s(f)\right\|_{\bar{q}} = J_1 + J_2. \tag{11}$$

We now estimate $J_2.$ Since $f \in H_{\bar{p}}^r,$ we have

$$\big\|\sigma_s(f)\big\|_{\bar{p}} \le 2^{-sr}, \quad s = 0, 1, \dots.$$

By the property of the norm, the inequality of different metrics (see Theorem B), and the inequality $r > m,$ we obtain

$$\begin{aligned} J_2 &= \left\|\sum_{s>[\alpha l]+1}\sigma_s(f)\right\|_{\bar{q}} \le \sum_{s>[\alpha l]+1}\big\|\sigma_s(f)\big\|_{\bar{q}} \\ &\ll \sum_{s>[\alpha l]+1} 2^{s\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\big\|\sigma_s(f)\big\|_{\bar{p}} \ll \sum_{s>[\alpha l]+1} 2^{s\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}2^{-sr} \\ &\ll 2^{-\alpha l\left(r-\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}. \end{aligned}$$

In view of the choice of the number $\alpha,$ we get

$$J_2 \ll 2^{-l\left(r-\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)} \asymp n^{-\left(\frac{r}{m}-\frac{1}{m}\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)}. \tag{12}$$

We now estimate J_1. For a natural number s satisfying the inequalities $l \le s \le [\alpha l]+1,$ we consider an operator T_s given by the formula

$$T_s f(\bar{x}) = f(\bar{x}) * \Big(t_s(\bar{x}) - t(\Omega_{n_s}, \bar{x})\Big).$$

Let $p_j \in (1,2),\ j=1,\dots,m$. By using the Minkowski inequality and Lemma 1, we get

$$J_1 = \left\| \sum_{s=l}^{[\alpha l]+1} \sigma_s(f,\bar{x}) - \Big(t(\Omega_{n_s};\bar{x}) * \sigma_s(f,\bar{x})\Big) \right\|_{\bar{q}}$$

$$\le \sum_{s=l}^{[\alpha l]+1} \Big\| \sigma_s(f,\bar{x}) - \Big(t(\Omega_{n_s};\bar{x}) * \sigma_s(f,\bar{x})\Big) \Big\|_{\bar{q}} \ll \sum_{s=l}^{[\alpha l]+1} \big\| T_s \sigma_s(f) \big\|_{\bar{q}}$$

$$\ll \sum_{s=l}^{[\alpha l]+1} \|T_s\|_{\bar{p}\to\bar{q}} \big\|\sigma_s(f)\big\|_{\bar{p}} \ll \sum_{s=l}^{[\alpha l]+1} 2^{sm} n_s^{-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} \big\|\sigma_s(f)\big\|_{\bar{p}}. \tag{13}$$

Substituting the values of the numbers n_s in (13), we obtain

$$J_1 \ll \sum_{s=l}^{[\alpha l]+1} 2^{sm} \left(2^{lr} 2^{sm\left(1-\frac{r}{m}\right)}\right)^{-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} \big\|\sigma_s(f)\big\|_{\bar{p}}$$

$$\ll 2^{-l\frac{r}{m}\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} \sum_{s=l}^{[\alpha l]+1} 2^{sm} 2^{-s\left(1-\frac{r}{m}\right)\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} \big\|\sigma_s(f)\big\|_{\bar{p}}$$

$$\ll 2^{-l\frac{r}{m}\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} \sum_{s=l}^{[\alpha l]+1} 2^{2sm} 2^{-s\left(1-\frac{r}{m}\right)\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} 2^{-sr}. \tag{14}$$

As a result of simple calculations, we conclude that

$$m - \left(1-\frac{r}{m}\right)\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right) - r = \left(1-\frac{r}{m}\right)\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right).$$

It follows from relation (14) that

$$J_1 \ll 2^{-l\frac{r}{m}\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} \sum_{s=l}^{[\alpha l]+1} 2^{s\left(1-\frac{r}{m}\right)\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)}.$$

Since $\dfrac{2}{p_j} - 1 > 0$ and $1 - \dfrac{r}{m} < 0,$ we get

$$J_1 \ll 2^{-l\frac{r}{m}\sum_{j=1}^{m}\left(\frac{1}{2}+\frac{1}{p'_j}\right)} 2^{l\left(1-\frac{r}{m}\right)\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)} = C2^{-lm\left(\frac{r}{m}-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)} \asymp n^{-\left(\frac{r}{m}-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)}.$$

Hence,

$$J_1 \ll n^{-\left(\frac{r}{m}-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)} \tag{15}$$

in the case where $p_j \in (1,2),\ j=1,\dots,m.$

It follows from inequalities (11), (12), and (15) that

$$\|f-t\|_{\bar{q}} \ll n^{-\left(\frac{r}{m}-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)}$$

in the case where $p_j \in (1,2),\ j=1,\dots,m,$ for any function $f \in H^r_{\bar{p}}$. Thus,

$$d_n^T(H^r_{\bar{p}}, L_{\bar{q}}) \ll n^{-\left(\frac{r}{m}-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)}$$

for $p_j \in (1,2),\ j=1,\dots,m.$ In view of the inclusion $B^r_{\bar{p},\theta} \subset H^r_{\bar{p}}$ [see (1)], we find

$$d_n^T(B^r_{\bar{p},\theta}, L_{\bar{q}}) \ll n^{-\left(\frac{r}{m}-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)\right)}$$

for $p_j \in (1,2),\ j=1,\dots,m.$

The theorem is proved.

Remark 3. In the case where $p_1 = \ldots = p_m = p$ and $q_1 = \ldots = q_m = q$, Theorem 2 gives the results obtained by A. S. Romanyuk and V. S. Romanyuk in [14].

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