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To the solution of one pseudo-Volterra integral equation

In this paper, we study a homogeneous singular integral Volterra equation of the second kind (pseudo-Volterra integral equation). The singularity of the integral equation is shown. Properties of its kernel are proved. The characteristic equation is constructed. It is shown that it really is a characteristic equation for the studied integral equation. The kernel estimates of its integral operator are established. Solvability issues of the corresponding non-homogeneous integral equation are also researched. The weight class of the uniqueness for its solution is determined. A weight class is also established for the right side of the nonhomogeneous equation under study. The weight class of the uniqueness for its solution is defined on the basis of estimates for the kernel of the integral operator of the equation.

Keywords: characteristic equation, kernel, integral operator, class of essentially bounded functions.

Introduction

We study the solvability of a pseudo-Volterra integral equation:

$$\varphi(t) + \int_0^t K_\omega(t,\tau)\varphi(\tau)d\tau = 0, \tag{1}$$

where the kernel $K_\omega(t,\tau)$ is representable as a sum:

$$K_\omega(t,\tau) = \sum_{i=1}^{4} K_\omega^{(i)}(t,\tau),$$

and

$$K_\omega^{(1)} = \frac{1}{2a\sqrt{\pi}} \cdot \frac{t^\omega + \tau^\omega}{(t-\tau)^{3/2}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\};$$

$$K_\omega^{(2)} = -\frac{1}{2a\sqrt{\pi}} \cdot \frac{t^\omega - \tau^\omega}{(t-\tau)^{3/2}} \cdot \exp\left\{-\frac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\right\};$$

$$K_\omega^{(3)} = -\frac{1}{a\sqrt{\pi}} \cdot \frac{1+\omega t^{\omega-1}}{(t-\tau)^{1/2}} \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\};$$

$$K_\omega^{(4)} = \frac{1}{a\sqrt{\pi}} \cdot \frac{1+\omega t^{\omega-1}}{(t-\tau)^{1/2}} \exp\left\{-\frac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\right\}.$$

This kind of integral equations arise in solving the following boundary value problem:

$$\frac{\partial u(x,t)}{\partial t} - a^2\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad \{(x,t)\,|\,0<x<t^\omega, t>0\};$$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0, \quad \frac{d\tilde{u}(t)}{dt} + \frac{\partial u}{\partial x}\Big|_{x=t^\omega} = 0,$$

where $\tilde{u}(t) = u(t^\omega, t), \omega > \frac{1}{2}$.

We will search for the solution of the integral equation (1) in the class of functions

$$t^{\frac{3}{2}-\omega} \cdot \varphi(t) \in L_\infty(0,\infty);$$

i.e.

$$\varphi(t) \in L_\infty(0, \infty; t^{\frac{3}{2}-\omega}).$$

The equation (1) can be written as:

$$\varphi(t) + \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} K_\omega(t,\tau)\varphi(\tau)d\tau = 0. \tag{2}$$

Volterra integral equations of this kind were considered in papers [1–3].

1 About properties of the kernel $K_\omega(t,\tau)$

We note that the kernel $K_\omega(t,\tau)$ has the properties:
1)$K_\omega(t,\tau) \geq 0$ and is continuous when $0 < \tau \leq t \leq \infty$;
2) $\lim_{t\to t_0} \int_{t_0}^t K_\omega(t,\tau)d\tau = 0,\ t_0 \geq \varepsilon > 0$;
3) $\lim_{t\to 0+} \int_0^t K_\omega(t,\tau)d\tau = 1$.
The singularity of the integral equation (1) is property 3 of kernel $K_\omega(t,\tau)$. We prove this property.
Lemma 1. If $\omega > \frac{1}{2}$, then

$$\lim_{t\to 0+} \int_0^t K_\omega^{(1)}(t,\tau)d\tau = 1.$$

Proof. We have:

$$\int_0^t K_\omega^{(1)}(t,\tau)d\tau = \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^\omega + \tau^\omega}{(t-\tau)^{3/2}} \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau =$$

$$= \frac{2}{\sqrt{\pi}} \cdot \int_0^t \left[\frac{t^\omega + \tau^\omega}{4a(t-\tau)^{3/2}} + \frac{\omega\cdot\tau^{\omega-1}}{2a\sqrt{t-\tau}}\right] \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau -$$

$$-\frac{1}{a\sqrt{\pi}} \cdot \int_0^t \frac{\omega\cdot\tau^{\omega-1}}{\sqrt{t-\tau}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau = J_1 - J_2.$$

In the integral J_1 we make a replacement

$$\frac{t^\omega + \tau^\omega}{2a(t-\tau)^{1/2}} = x.$$

As a result, we get

$$J_1 = \frac{2}{\sqrt{\pi}} \cdot \int_{\frac{t^{\omega-\frac{1}{2}}}{2a}}^\infty \exp(-x^2)dx = 1 - erf\left\{\frac{t^{\omega-\frac{1}{2}}}{2a}\right\}.$$

We estimate the second integral J_2:

$$J_2 = \frac{1}{a\sqrt{\pi}} \cdot \int_0^t \frac{\omega\cdot\tau^{\omega-1}}{\sqrt{t-\tau}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2\,(t-\tau)}\right\} d\tau \leq$$

$$\leq \frac{\omega\cdot\tau^{\omega-\frac{1}{2}}}{a\sqrt{\pi}} \cdot \int_0^t \frac{d\tau}{\sqrt{\tau\cdot(t-\tau)}} = \frac{\omega\cdot\sqrt{\pi}}{a} t^{\omega-\frac{1}{2}}.$$

As from conditions $\omega > \dfrac{1}{2}$ and $t \to 0$ it follows $erf\left\{\dfrac{t^{\omega-\frac{1}{2}}}{2a}\right\} \to 0$, then from here we get the required ratio.

Lemma 2. If $\omega > \dfrac{1}{2}$, then

$$\lim_{t\to 0} \int_0^t K_\omega^{(i)}(t,\tau)d\tau = 0, \quad i = 2,3,4.$$

Proof. Let be $\dfrac{1}{2} < \omega \leq 1$. In this case, we will have at $t \to 0$:

$$0 \leq \int_0^t K_\omega^2(t,\tau)d\tau \leq \frac{\omega}{2a\sqrt{\pi}} \cdot \int_0^t \frac{\tau^{\omega-1}}{\sqrt{t-\tau}}d\tau = \frac{\omega}{2a\sqrt{\pi}} \cdot B\left(\omega, \frac{1}{2}\right) \cdot t^{\omega-\frac{1}{2}} \to 0.$$

In this case, we have used the following double inequality [4; 55] for all $0 < \tau \le t \le \infty$:

$$\omega \cdot t^{\omega-1}(t-\tau) \le t^{\omega} - \tau^{\omega} \le \omega \cdot \tau^{\omega-1}(t-\tau).$$

at $0 < \omega \le 1$.

Now let be $\omega > 1$. Then we have:

$$\int_0^t K_\omega^2(t,\tau)d\tau \le \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^\omega - \tau^\omega}{(t-\tau)^{3/2}} d\tau =$$

$$= \left\| \tau = t\cdot \sin^2\alpha,\ d\tau = 2t\cdot\sin\alpha\cdot\cos\alpha d\alpha \right\| =$$

$$= \frac{2\cdot t^{\omega-\frac12}}{2a\sqrt{\pi}} \cdot \int_0^{\frac{\pi}{2}} \frac{\sin\alpha(1-\sin^{2\omega}\alpha)}{\cos^{\frac12}\alpha} d\alpha = C_1(\omega)\frac{2\cdot t^{\omega-\frac12}}{2a\sqrt{\pi}}.$$

The last integral is bounded (i.e. a number $C_1(\omega)$ is bounded) due to the existence of a finite limit for the integrand at a unique singular point $\alpha = \frac{\pi}{2}$. Indeed, calculating the next limit:

$$\lim_{\alpha\to\frac{\pi}{2}} \frac{\sin\alpha(1-\sin^{2\omega}\alpha)}{\cos^{\frac12}\alpha} = \lim_{\alpha\to\frac{\pi}{2}} \frac{(1-\sin^{2\omega}\alpha)}{\cos^{\frac12}\alpha} = 4\omega\cdot \lim_{\alpha\to\frac{\pi}{2}} \sin^{2\omega-1}\alpha\cdot\cos^{\frac32}\alpha = 0,$$

we obtain at $\omega > 1$ required limit

$$\lim_{t\to 0}\int_0^t K_\omega^2(t,\tau)d\tau = 0.$$

For the kernels $K_\omega^3(t,\ \tau)$, $K_\omega^4(t,\tau)$ the proof is obvious.

Lemma 2 is proved.

In the sequel, we need the following lemmas.

Lemma 3. If $\omega > \frac12$, then

$$t^{\frac32-\omega}\int_0^t \frac{K_\omega^{(1)}(t,\tau)}{\tau^{\frac32-\omega}} d\tau < C = const.\ 0 < t < \infty.$$

Proof. Indeed, we have:

$$t^{\frac32-\omega}\int_0^t \frac{K_\omega^{(1)}(t,\tau)}{\tau^{\frac32-\omega}} d\tau = \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^{\frac32-\omega}}{\tau^{\frac32-\omega}} \frac{t^\omega + \tau^\omega}{(t-\tau)^{\frac32}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau \le$$

$$\le \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \frac{t^{\frac32-\omega}}{\tau^{\frac32-\omega}} \frac{2t^\omega}{(t-\tau)^{\frac32}} \cdot \exp\left\{-\frac{t^{2\omega}}{4a^2(t-\tau)}\right\} d\tau \le$$

$$\le \left\| \frac{\tau}{t} = 1-x;\quad \frac{t^\omega}{2a\sqrt{t-\tau}} = \frac{t^{\omega-\frac12}}{2a\sqrt{x}};\ \frac{t^\omega dx}{4a(t-\tau)^{\frac32}} = -\frac{t^{\omega-\frac12}dx}{4ax^{\frac32}} \right\| =$$

$$= \frac{1}{a\sqrt{\pi}} \int_0^1 t^{\omega-\frac12} \cdot \frac{(1-x)^{\omega-\frac32}}{x^{\frac32}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx =$$

$$= \frac{t^{\omega-\frac12}}{a\sqrt{\pi}} \left[\int_0^{\frac12} \frac{(1-x)^{\omega-\frac32}}{x^{\frac32}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx + \int_{\frac12}^1 \frac{(1-x)^{\omega-\frac32}}{x^{\frac32}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx\right] \le$$

$$\le \frac{C(\omega)}{a\sqrt{\pi}} \int_0^{\frac12} \frac{t^{\omega-\frac12}}{x^{\frac32}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2x}\right\} dx + \frac{2^{\frac32}t^{\omega-\frac12}}{a\sqrt{\pi}} \int_{\frac12}^1 (1-x)^{\omega-\frac32} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2}\right\} dx \equiv J,$$

where

$$C(\omega) = \begin{cases} 1, & \omega \ge \frac32, \\ 2^{\frac32-\omega}, & \frac12 \le \omega < \frac32, \end{cases} \quad \max_{\frac12<\omega<\infty} C(\omega) = 2.$$

Further, if we make the following replacement in the first integral:

$$\xi = \frac{t^{\omega-\frac{1}{2}}}{2a\sqrt{x}}, \ \ d\xi = -\frac{t^{\omega-\frac{1}{2}}}{4ax^{\frac{3}{2}}}dx$$

and leave the second integral unchanged, then we will have

$$J = \frac{8}{\pi}\int_{\frac{t^{\omega-\frac{1}{2}}}{a\sqrt{2}}}^{\infty} \exp\left\{-\xi^2\right\} dx - \frac{2^{\frac{3}{2}}t^{\omega-\frac{1}{2}}}{a\sqrt{\pi}} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2}\right\} \cdot \left.\frac{(1-x)^{\omega-\frac{1}{2}}}{\omega-\frac{1}{2}}\right|_{\frac{1}{2}}^{1} =$$

$$= 4\, erfc\left(\frac{t^{\omega-\frac{1}{2}}}{a\sqrt{2}}\right) + \frac{2^{4-\omega}}{a(2\omega-1)} \cdot \frac{t^{\omega-\frac{1}{2}}}{2a} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2}\right\} \le C < \infty.$$

Hence the assertion of Lemma 3 follows.

Lemma 4. If $\omega > \frac{1}{2}$, then

$$t^{\frac{3}{2}-\omega}\int_0^t \frac{K_\omega^{(2)}(t,\tau)}{\tau^{\frac{3}{2}-\omega}}d\tau < C(\omega) = const. \quad 0 < t < \infty.$$

Proof. We have:

$$t^{\frac{3}{2}-\omega}\int_0^t \frac{K_\omega^{(2)}(t,\tau)}{\tau^{\frac{3}{2}-\omega}}d\tau = \frac{1}{2a\sqrt{\pi}} \cdot \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} \frac{t^\omega - \tau^\omega}{(t-\tau)^{\frac{3}{2}}} \cdot \exp\left\{-\frac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\right\} d\tau \le$$

$$\le \frac{1}{\sqrt{\pi}}\left\{\frac{(1-x^\omega)|_{x=0}}{(1-x)^{\frac{3}{2}}\Big|_{x=1}}\left[\frac{t^{\omega-\frac{1}{2}}}{2a} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2} \cdot \frac{(1-x^\omega)^2\big|_{x=\frac{1}{2}}}{(1-x)|_{x=0}}\right\}\right]\int_0^{\frac{1}{2}} x^{\omega-\frac{3}{2}}dx + \right.$$

$$\left. + \sup_{\frac{1}{2}<x<1}\left\{x^{\omega-\frac{3}{2}}\right\}\left[\frac{t^{\omega-\frac{1}{2}}}{2a} \cdot \exp\left\{-\frac{t^{2\omega-1}}{4a^2} \cdot \frac{(1-\overline{x}^\omega)^2}{1-\overline{x}}\right\}\right]\int_{\frac{1}{2}}^1 \frac{1-x^\omega}{(1-x)^{\frac{3}{2}}}d\,x\right\} \le \ C(\omega) = const.$$

For the kernels $K_\omega^3(t,\,\tau)$, $K_\omega^4(t,\tau)$ the proof is obvious.

Lemma 4 is proved.

Remark 1. Since, according to Lemma 1, there is the singularity of the kernel $K_\omega^{(1)}(t,\tau)$, for that the statement of Lemma 3 holds, then for the kernel $K_\omega^{(2)}(t,\tau)$ with a weak singularity (Lemma 2), the statement of the Lemma 4 becomes obvious. This has showed the above proof of the Lemma 4.

2 Characteristic integral equation. Estimates for the kernels of integral operators

For the integral equation (2) we will construct a characteristic equation

$$\varphi(t) + \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} K_h(t,\tau) \cdot \varphi(\tau)\, d\tau = g(t), \tag{3}$$

where

$$K_h(t,\tau) = \sum_{i=1}^{4} K_h^{(i)}(t,\tau);$$

$$K_h^{(1)}(t,\tau) = \frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2}\left(\tau^{2\omega-1} \cdot t^{2\omega-2} + t^{4\omega-3}\right)}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{3/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}+\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$

$$K_h^{(2)}(t,\tau) = -\frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2} \cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$

$$K_h^{(3)}(t,\tau) = -\frac{2}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2} \cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}+\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$

$$K_h^{(4)}(t,\tau) = \frac{2}{a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{3/2} \cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}} \cdot \exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\}.$$

Let us show that it really is a characteristic equation for the equation (1). First, we note that the kernel $K_h(t,\tau)$ also has a property similar to the property 3 of the kernel $K_\omega(t,\tau)$

$$\lim_{t\to 0}\int_0^t K_h^{(1)}(t,\tau)\,d\tau = 1.$$

Equation (3) with the following replacements:

$$t = \left(\frac{1}{2\omega-1}\cdot t_1\right)^{\frac{1}{2\omega-1}}, \tau = \left(\frac{1}{2\omega-1}\cdot \tau_1\right)^{\frac{1}{2\omega-1}};$$

$$\varphi\left[\left(\frac{1}{2\omega-1}\cdot t_1\right)^{\frac{1}{2\omega-1}}\right] = \varphi_1(t_1), g\left[\left(\frac{1}{2\omega-1}\cdot t_1\right)^{\frac{1}{2\omega-1}}\right] = g_1(t_1),$$

is reduced to an integral equation of the form:

$$\varphi_1(t_1) + \int_0^{t_1}\sqrt{\frac{t_1}{\tau_1}}\cdot K_1(t_1,\tau_1)\cdot\varphi_1(\tau_1)\,d\tau_1 = g_1(t_1). \tag{4}$$

The kernel $K_1(t,\tau)$ has the form:

$$K_1(t,\tau) = \sum_{i=1}^{4} K_1^{(i)}(t,\tau),$$

where

$$K_1^{(1)}(t,\tau) = \frac{1}{2a\sqrt{\pi}}\cdot\frac{t+\tau}{(t-\tau)^{3/2}}\cdot\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\};$$

$$K_1^{(2)}(t,\tau) = -\frac{1}{2a\sqrt{\pi}}\cdot\frac{t-\tau}{(t-\tau)^{3/2}}\cdot\exp\left\{-\frac{(t-\tau)^2}{4a^2(t-\tau)}\right\};$$

$$K_1^{(3)}(t,\tau) = -\frac{2}{a\sqrt{\pi}}\cdot\frac{1}{(t-\tau)^{1/2}}\cdot\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\};$$

$$K_1^{(4)}(t,\tau) = \frac{2}{a\sqrt{\pi}}\cdot\frac{1}{(t-\tau)^{1/2}}\cdot\exp\left\{-\frac{(t-\tau)^2}{4a^2(t-\tau)}\right\}. \tag{5}$$

We found a solution to the equation (4) with kernel (5) and it has the form [5]:

$$\varphi(t) = g(t) + \int_0^t\sqrt{\frac{t}{\tau}}\cdot R(t,\tau)\cdot g(t)\,d\tau_1 + C\cdot\varphi_0(t),$$

where the resolvent $R(t,\tau) = R_1(t,\tau) + R_2(t,\tau)$ has the form:

$$R_1(t,\tau) = \frac{1}{a\sqrt{\pi}\,(\tau-t)^{\frac{3}{2}}\,\tau}\cdot\sum_{n=0}^{\infty}(-1)^n B_n\cdot\left[n\cdot\exp\left\{-\frac{n^2}{a^2(\tau-t)}\right\} + 3(n+1)\cdot\exp\left\{-\frac{(n+1)^2}{a^2(\tau-t_1)}\right\}+\right.$$

$$+3\,(n+2)\cdot\exp\left\{-\frac{(n+2)^2}{a^2\,(\tau-t)}\right\}+(n+3)\cdot\exp\left\{-\frac{(n+3)^2}{a^2\,(\tau-t_1)}\right\}\Bigg];$$

$$R_2(t,\tau)=\frac{3}{2a\sqrt{\pi}}\frac{1}{\tau\sqrt{\tau-t}\,(2\tau-t)}+$$

$$+\frac{3}{2a^2\pi\tau}\sum_{n=1}^{\infty}(-1)^n B_n\cdot\left[r_n\,(t,\tau)+r_{n+1}\,(t,\tau)-r_{n+2}\,(t,\tau)-r_{n+3}\,(t,\tau)\right];$$

$$\varphi_0\,(t)=\frac{C}{2\sqrt{\pi}}\cdot\sum_{n=1}^{\infty}(-1)^n\,(2n+3)\,B_n\cdot\exp\left\{-\frac{(2n+3)^2}{4a^2}\cdot t\right\}.$$

For the resolvent the following estimate is valid:

$$|R(t,\tau)|\leq\frac{1}{t\cdot\tau^2}\cdot\left[\frac{\tau\cdot\tau^{\frac{3}{2}}\cdot t^{\frac{3}{2}}}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2\,(t-\tau)}\right\}+\frac{\tau\cdot\sqrt{\tau}\cdot\sqrt{t}\cdot t\cdot\tau}{\sqrt{t-\tau}\,(2t-\tau)}\right]\exp\left\{-\frac{t-\tau}{4a^2}\right\}\leq$$

$$\leq C_3\left[\frac{\sqrt{\tau}\cdot\sqrt{t}}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2\,(t-\tau)}\right\}+\frac{\sqrt{\tau}\cdot\sqrt{t}}{\sqrt{t-\tau}\,(2t-\tau)}\right]\exp\left\{-\frac{t-\tau}{4a^2}\right\}.$$

Returning to the old variables, i.e. making replacements

$$\tau_1=(2\omega-1)\cdot\tau^{2\omega-1},\quad \varphi_1\left[(2\omega-1)\cdot t^{2\omega-1}\right]=\varphi\,(t);$$

$$t_1=(2\omega-1)\cdot t^{2\omega-1},\quad g_1\left[(2\omega-1)\cdot t^{2\omega-1}\right]=g\,(t),$$

we get the solution of the characteristic equation (3):

$$\varphi\,(t)=g(t)+\int_0^t\left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega}\cdot R_h(t,\tau)\cdot g(t)d\tau+C\cdot\varphi_0((2\omega-1)\cdot t^{2\omega-1}),$$

where the resolvent $R_h\,(t,\tau)$ will have the following estimate

$$R_h(t,\tau)\leq C_2(\omega)\left[\left(\frac{1}{2\omega-1}\right)^{\frac{3}{2}}\cdot\frac{t^{\omega-\frac{1}{2}}\tau^{\omega-\frac{1}{2}}\tau^{2\omega-2}}{(t^{2\omega-1}-\tau^{2\omega-1})}\exp\left\{-(2\omega-1)\frac{t^{2\omega-1}\cdot\tau^{2\omega-1}}{a^2\,(t^{2\omega-1}-\tau^{2\omega-1})}\right\}+\right.$$

$$\left.+(2\omega-1)^{\frac{3}{2}}\cdot\frac{t^{\omega-\frac{1}{2}}\tau^{\omega-\frac{1}{2}}\tau^{2\omega-2}}{(t^{2\omega-1}-\tau^{2\omega-1})^{\frac{1}{2}}\,(2t^{2\omega-1}-\tau^{2\omega-1})}\right]\leq$$

$$\leq C_3(\omega)\,(2\omega-1)^{\frac{3}{2}}\left[\left(\frac{1}{2\omega-1}\right)^{\frac{3}{2}}\cdot\frac{t^{\omega-\frac{1}{2}}\cdot\tau^{3\omega-\frac{5}{2}}}{\tau^{3\omega-3}(t-\tau)^{\frac{3}{2}}}\exp\left\{\frac{t^{2\omega-1}\cdot\tau}{a^2\,(t-\tau)}\right\}+\frac{t^{\omega-\frac{1}{2}}\cdot\tau^{3\omega-\frac{5}{2}}}{(2\omega-1)^{\frac{1}{2}}\,(t-\tau)^{\frac{1}{2}}\tau^{2\omega-\frac{3}{2}}}\right]\leq$$

$$\leq C_3(\omega)\frac{\sqrt{\tau}\cdot t^{\omega-\frac{1}{2}}}{(t-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{a^2(t-\tau)}\right\}+C_4(\omega)\frac{\tau^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau}}.$$

Indeed:

$$\frac{t^{\omega-\frac{1}{2}}\cdot\tau^{3\omega-\frac{5}{2}}}{(2\omega-1)^{\frac{1}{2}}\,(t-\tau)^{\frac{1}{2}}\tau^{2\omega-\frac{3}{2}}}\leq C_4(\omega)\frac{\tau^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau}}.$$

Theorem 1. The general solution to the characteristic integral equation has the form

$$\varphi\,(t)=g(t)+\int_0^t\left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega}\cdot R_h(t,\tau)\cdot g(t)d\tau+C\cdot\varphi_0((2\omega-1)\cdot t^{2\omega-1}).$$

3 Solution of an integral equation.
Regularization method by the solution of the characteristic equation

Remark 2 [6; 183]. If the (particular) solution of the integral equation

$$y(x)+\int_a^x K\left(x,t\right)y\left(t\right)dt=f\left(x\right)$$

is given by the formula

$$y(x)=f\left(x\right)+\int_a^x R\left(x,t\right)f\left(t\right)dt$$

then the (particular) solution of the integral equation (with a modified kernel)

$$y(x)+\int_a^x K\left(x,t\right)\frac{g\left(x\right)}{g\left(t\right)}y\left(t\right)dt=f\left(x\right)$$

is given by the formula

$$y(x)=f\left(x\right)+\int_a^x R\left(x,t\right)\frac{g\left(x\right)}{g\left(t\right)}y\left(t\right)dt.$$

The same is true for solutions of the corresponding homogeneous equations.

Using Remark 2, we consider equation (1), which we represent in the form

$$\varphi(t)+\int_0^t K_h(t,\tau)\cdot\varphi(\tau)d\tau=\int_0^t\left[K_h(t,\tau)-K_\omega(t,\tau)\right]\cdot\varphi(\tau)d\tau. \tag{6}$$

Assuming the right side of the equation (6) is temporarily known, we write its solution:

$$\varphi(t)=\int_0^t\left[K_h(t,\tau)-K_\omega(t,\tau)\right]\cdot\varphi(\tau)d\tau+\int_0^t R_\omega(t,\tau)\cdot\left\{\int_0^\tau\left[K_h(\tau,\tau_1)-K_\omega(\tau,\tau_1)\right]\cdot\varphi(\tau_1)d\tau_1\right\}d\tau+$$
$$+C_0\cdot\varphi_0\left((2\omega-1)\cdot t^{2\omega-1}\right).$$

In the repeated integral, we change the order of integration and change roles by variables τ and τ_1, we obtain

$$\varphi(t)+\int_0^t\bar{\tilde{K}}(t,\tau)\cdot\varphi(\tau)d\tau=C\cdot\varphi_0\left((2\omega-1)\cdot t^{2\omega-1}\right), \tag{7}$$

the kernel $\bar{\tilde{K}}(t,\tau)$ has the form

$$\bar{\tilde{K}}(t,\tau)=\tilde{K}(t,\tau)+\bar{K}(t,\tau), \tag{8}$$

where

$$\tilde{K}(t,\tau)=K_h(t,\tau)-K_\omega(t,\tau),\ \bar{K}(t,\tau)=\int_\tau^t R(t,\tau_1)\cdot\left[K_h(\tau_1,\tau)-K_\omega(\tau_1,\tau)\right]d\tau_1.$$

First we estimate the function $\tilde{K}(t,\tau)$, that is the first term of (8). For this we introduce the following notation:

$$K_h^{(i)}(t,\tau)=P_h^{(i)}e^{-Q_h^{(i)}};\ K_\omega^{(i)}(t,\tau)=P_\omega^{(i)}e^{-Q_\omega^{(i)}},\quad i=1,2,3,4,$$

where

$$P_h^1(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}(\tau^{2\omega-1}\cdot t^{2\omega-2}+t^{4\omega-3})}{(t^{2\omega-1}-\tau^{2\omega-1})^{\frac{3}{2}}};$$

$$P_\omega^{(1)}(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{t^\omega+\tau^\omega}{(t-\tau)^{\frac{3}{2}}};$$

$$Q_h^{(1)}(t,\tau)=\exp\left\{-\frac{(2\omega-1)(t^{2\omega-1}+\tau^{2\omega-1})^2}{4a^2(t^{2\omega-1}-\tau^{2\omega-1})}\right\};$$

$$Q_\omega^{(1)}(t,\tau)=\exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2(t-\tau)}\right\}.$$

The following statement is true.

Lemma 5. If $\omega > \frac{1}{2}$, then

$$\lim_{t\to 0}\int_0^t \tilde{K}(t,\tau)d\tau = 0,\ \ 0<\tau<t<\infty \tag{9}$$

and the following estimate is correct

$$\left|\tilde{K}(t,\tau)\right| \le C_1(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}e^{-\tilde{Q}(t,\tau)} + C_2(\omega)\frac{1}{\sqrt{t-\tau}}, \tag{10}$$

where $\tilde{Q}(t,\tau) = \min\left\{Q_h(t,\tau); \frac{1}{2}Q^1_\omega(t,\tau)\right\}$.

Proof. We note that from the estimate (10) immediately follows the ratio (9). Note that the estimate (10) obvious for summands

$$\left|K_h^{(i)}(t,\tau) - K_\omega^{(i)}(t,\tau)\right| \ \text{ at } \ i = 2,3,4.$$

We prove the estimate (10) for the first summand $\left|K^1_h(t,\tau) - K^1_\omega(t,\tau)\right|$.

We have:

$$\left|K_h^{(1)}(t,\tau) - K_\omega^{(1)}(t,\tau)\right| =$$

$$= \left|P_h^{(1)}(t,\tau)\exp\left\{-Q_h^{(1)}(t,\tau)\right\} - P_\omega^{(1)}(t,\tau)\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\}\right| \le$$

$$\le \left|P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_h^{(1)}(t,\tau)\right\} +$$

$$+P_\omega^{(1)}(t,\tau)\exp\left\{-Q_h^{(1)}(t,\tau)\right\}\ \left|1-\exp\left\{-Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right\}\right| \le$$

$$\le \left|P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_h^{(1)}(t,\tau)\right\} +$$

$$+P_\omega^{(1)}(t,\tau)\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\}.$$

For further calculations, we first prove the following lemma.

Lemma 6. There are relationships:

$$\left|P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right| \le C_2(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}};$$

$$P_\omega^{(1)}(t,\tau)\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\} \le$$

$$\le C_3(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}\exp\left\{-\frac{Q_\omega^{(1)}(t,\tau)}{2}\right\}. \tag{11}$$

Proof. We introduce the notation:

$$\tilde{P}^1(t,\tau) = \left[P_h^{(1)}(t,\tau) - P_\omega^{(1)}(t,\tau)\right]\cdot\frac{\sqrt{t-\tau}}{t^{\omega-1}}.$$

Then, making a replacement $\tau = tx$, $(0<x<1)$, we obtain

$$\tilde{P}(t,tx) = \frac{1}{2a\sqrt{\pi}}\left((2\omega-1)^{\frac{3}{2}}\cdot\frac{t^{4\omega-3}+t^{2\omega-2}\tau^{2\omega-1}}{(t^{2\omega-1}-\tau^{2\omega-1})^{\frac{3}{2}}} - \frac{t^\omega+\tau^\omega}{(t-\tau)^{\frac{3}{2}}}\right)\cdot\frac{\sqrt{t-\tau}}{t^{\omega-1}} =$$

$$= \frac{1}{2a\sqrt{\pi}}\left((2\omega-1)^{\frac{3}{2}}\cdot\frac{1+x^{2\omega-1}}{(1-x^{2\omega-1})^{\frac{3}{2}}} - \frac{1+x^\omega}{(1-x)^{\frac{3}{2}}}\right)\cdot\sqrt{1-x} =$$

$$= \frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)^{\frac{3}{2}}(1+x)^{2\omega-1}(1-x)^{\frac{3}{2}} - (1+x^{\omega})(1-x^{2\omega-1})^{\frac{3}{2}}}{(1-x)(1-x^{2\omega-1})^{\frac{3}{2}}}.$$

$$\lim_{x\to 1} \frac{1}{2a\sqrt{\pi}} \cdot \frac{1+x^{2\omega-1}-1-x^{\omega}}{1-x} = \frac{1}{2a\sqrt{\pi}} \cdot \frac{(2\omega-1)\cdot x^{2\omega-1} - \omega \cdot x^{\omega-1}}{-1} = \frac{\omega-1}{2a\sqrt{\pi}}.$$

Let us show the validity of the second statement (11) of Lemma 2. First of all, we note that

$$\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right| \le C_4(\omega)\,|1-\omega|\,t^{2\omega-1}. \tag{12}$$

Indeed, we have

$$t^{1-2\omega}\,|Q_h(t,\tau)-Q_\omega(t,\tau)| = \frac{t^{1-2\omega}}{4a^2}\cdot\left\{(2\omega-1)\frac{(t^{2\omega-1}+\tau^{2\omega-1})^2}{t^{2\omega-1}-\tau^{2\omega-1}} - \frac{(t^{\omega}+\tau^{\omega})^2}{t-\tau}\right\} =$$

$$= \frac{t^{1-2\omega}}{4a^2}\cdot\left\{(2\omega-1)(t^{2\omega-1}-\tau^{2\omega-1}) - \frac{(t^{\omega}-\tau^{\omega})^2}{t-\tau} + \frac{4(2\omega-1)t^{2\omega-1}\tau^{2\omega-1}}{t^{2\omega-1}-\tau^{2\omega-1}} - \frac{4t^{\omega}\tau^{\omega}}{t-\tau}\right\} =$$

$$= \|\tau = tx\| = \frac{1}{4a^2}\left\{(2\omega-1)(1-x^{2\omega-1}) - \frac{(1-x^{\omega})^2}{1-x} + \frac{4(2\omega-1)x^{2\omega-1}}{1-x^{2\omega-1}} - \frac{4x^{\omega}}{1-x}\right\}.$$

$$\lim_{x\to 1}\frac{1}{a^2}\left\{\frac{(2\omega-1)x^{2\omega-1}(1-x) - x^{\omega}(1-x^{2\omega-1})}{(1-x)(1-x^{2\omega-1})}\right\} =$$

$$= \left\|1-x^{2\omega-1} \approx (2\omega-1)(1-x)\; if\; x\to 1\right\| = \frac{1}{a^2}\frac{(2\omega-1)[x^{2\omega-1}-x^{\omega}]}{1-x^{2\omega-1}}.$$

So,

$$t^{1-2\omega}\,|Q_h(t,\tau)-Q_\omega(t,\tau)| =$$

$$= \begin{cases} \dfrac{2\omega-1}{a^2}\cdot x^{2\omega-1}\dfrac{1-x^{1-\omega}}{1-x^{2\omega-1}} \approx \dfrac{1-\omega}{a^2},\ \dfrac{1}{2}<\omega<1; \\ \dfrac{2\omega-1}{a^2}\cdot x^{2\omega-1}\dfrac{1-x^{1-\omega}}{1-x^{2\omega-1}} \approx -\dfrac{\omega-1}{a^2},\ \omega>1. \end{cases}$$

This directly implies the inequality (12).

We now turn to the proof of inequality (11). We have:

$$P_\omega^{(1)}(t,\tau)\left|Q_h^{(1)}(t,\tau) - Q_\omega^{(1)}(t,\tau)\right|\exp\left\{-Q_\omega^{(1)}(t,\tau)\right\} \le$$

$$\le C_4(\omega)\cdot t^{2\omega-1}\frac{t^{\omega}+\tau^{\omega}}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{(t^{\omega}+\tau^{\omega})^2}{4a^2(t-\tau)}\right\} \le$$

$$\le C_5(\omega)\cdot t^{2\omega-1}\frac{t^{\omega}+\tau^{\omega}}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\cdot\frac{8a^2(t-\tau)}{(t^{\omega}+\tau^{\omega})^2}\exp\left\{-\frac{(t^{\omega}+\tau^{\omega})^2}{8a^2(t-\tau)}\right\} =$$

$$= C_5(\omega)\cdot t^{2\omega-1}\frac{1}{\sqrt{\pi}(t-\tau)^{\frac{1}{2}}}\cdot\frac{4a}{t^{\omega}+\tau^{\omega}}\exp\left\{-\frac{(t^{\omega}+\tau^{\omega})^2}{8a^2(t-\tau)}\right\} \le$$

$$\le C_3(\omega)\cdot\frac{t^{\omega-1}}{(t-\tau)^{\frac{1}{2}}}\cdot\exp\left\{-\frac{Q_\omega(t,\tau)}{2}\right\}.$$

Lemma 6 is completely proved.

The proof of Lemma 5 is completed by applying the estimates from Lemma 6.

Lemma 7. If $\omega > 1/2$, *then the following estimate is correct*

$$|\overline{K}(t,\tau)| \le C\left[t^{2w-2} + t^{w-1} + \frac{1}{\sqrt{t-\tau}}\cdot\exp\left\{-\frac{t^{2w-1}\cdot\tau}{t-\tau}\right\} + \frac{t^{w-1}}{\sqrt{t-\tau}}\cdot\exp\left\{-\frac{t^{2w-1}\cdot\tau}{t-\tau}\right\}\right].$$

We now turn to an estimate of the summand $\overline{K}(t,\tau)$ from (8).

$$\overline{K}(t,\tau) = \int_\tau^t R_h(t,\tau_1)\cdot\tilde{K}(\tau_1,\tau)d\tau_1,$$

using the following inequalities:

$$R_h(t,\tau) \le C_5(\omega)\frac{\sqrt{\tau}\cdot t^{\omega-\frac{1}{2}}}{(t-\tau)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{a^2(t-\tau)}\right\} + C_6(\omega)\frac{\tau^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau}};$$

$$|\tilde{K}(t,\tau)| \le C_1(\omega)\frac{t^{\omega-1}}{\sqrt{t-\tau}}\exp\left\{-\tilde{Q}(t,\tau)\right\} + C_2(\omega)\frac{1}{\sqrt{t-\tau}}.$$

The estimate is carried out in four stages:

1) $I_{11} = \int_\tau^t \frac{\tau_1^{\omega-1}}{\sqrt{\tau_1-\tau}}\cdot\frac{t^{\omega-1/2}\sqrt{\tau_1}}{(t-\tau_1)^{3/2}}\cdot\exp\left\{-\frac{t^{2\omega-1}\cdot\tau_1}{a^2(t-\tau_1)}\right\}d\tau_1 =$

$= t^{\omega-\frac{1}{2}}\int_0^\infty \frac{(1+z^2)^{\frac{1}{2}}}{(t-\tau)^{1/2}}\cdot\frac{(1+z^2)^{3/2}}{(t-\tau)^{3/2}}\left(\frac{tz^2+\tau}{1+z^2}\right)^{\omega-\frac{1}{2}}\frac{(t-\tau)2z}{(1+z^2)^2}\exp\left\{-\frac{t^{2\omega-1}\cdot(tz^2+\tau)}{t-\tau}\right\}dz =$

$= \frac{2\cdot t^{\omega-\frac{1}{2}}}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\int_0^\infty t^{\omega-\frac{1}{2}}\cdot\left(\frac{z^2+\frac{\tau}{t}}{1+z^2}\right)^{\omega-\frac{1}{2}}\exp\left\{-\frac{t^{2\omega}\cdot z^2}{a^2(t-\tau)}\right\}dz \le$

$\le \frac{2\cdot t^{2\omega-1}}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\frac{\sqrt{\pi}}{2}\frac{a\sqrt{t-\tau}}{t^\omega} = \frac{a\sqrt{\pi}t^{\omega-1}}{\sqrt{t-\tau}}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}.$

Further

2) $I_{12} = \int_\tau^t \frac{1}{\sqrt{\tau_1-\tau}}\frac{t^{\omega-1/2}\sqrt{\tau_1}}{(t-\tau_1)^{3/2}}\exp\left\{-\frac{t^{2\omega-1}}{a^2(t-\tau_1)}\right\}d\tau_1 =$

$= t^{\omega-\frac{1}{2}}\int_0^\infty \frac{(1+z^2)^{\frac{1}{2}}}{(t-\tau)^{1/2}}\frac{(1+z^2)^{3/2}}{(t-\tau)^{3/2}}\left(\frac{tz^2+\tau}{1+z^2}\right)^{1/2}\frac{(t-\tau)\cdot 2z}{(1+z^2)^2}\exp\left\{-\frac{t^{2\omega-1}\cdot(tz^2+\tau)}{t-\tau}\right\}dz =$

$= \frac{2\cdot t^{\omega-1/2}}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\int_0^\infty \sqrt{t}\cdot\left(\frac{z^2+\tau/t}{1+z^2}\right)^{1/2}\exp\left\{-\frac{t^{2\omega}\cdot z^2}{a^2(t-\tau)}\right\}dz \le$

$\le \frac{2\cdot t^\omega}{t-\tau}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}\frac{\sqrt{\pi}}{2}\cdot\frac{a\sqrt{t-\tau}}{t^\omega} = \frac{a\sqrt{\pi}}{\sqrt{t-\tau}}\exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{t-\tau}\right\}.$

3) $I_{21} = \int_\tau^t \frac{\tau_1^{\omega-1}}{\sqrt{\tau_1-\tau}}\cdot\frac{\tau_1^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau_1}}d\tau_1 =$

$$= \left\| \begin{array}{l} \dfrac{\tau_1-\tau}{t-\tau_1} = z^2;\quad \tau_1-\tau = (t-\tau_1)z^2;\ \tau_1 = \dfrac{tz^2+\tau}{1+z^2}; \\ \tau_1-\tau = (t-\tau)\dfrac{z^2}{1+z^2};\ t-\tau_1 = \dfrac{t-\tau}{1+z^2};\quad d\tau_1 = \dfrac{2(t-\tau)zdz}{(1+z^2)^2} \end{array} \right\| =$$

$= t^{-\omega-1/2}\int_0^\infty \left(\frac{tz^2+\tau}{1+z^2}\right)^{3\omega-3/2}\frac{1+z^2}{(t-\tau)z}\cdot\frac{(t-\tau)\cdot 2z\cdot dz}{(1+z^2)^2} \le C\cdot t^{2\omega-2}.$

4) $I_{22} = \int_\tau^t \frac{1}{\sqrt{\tau_1-\tau}}\cdot\frac{\tau_1^{2\omega-\frac{1}{2}}}{t^{\omega+\frac{1}{2}}\cdot\sqrt{t-\tau_1}}d\tau_1 = t^{-\omega-\frac{1}{2}}\int_0^\infty\left(\frac{1+z^2}{tz^2+\tau}\right)^{2\omega-\frac{1}{2}}\frac{2dz}{1+z^2} \le C\cdot t^{\omega-1}.$

Lemma is iproved.

4 Main result

Thus, the following statement is proved:

Theorem 2. If $\omega > \frac{1}{2}$, then the kernel of the integral equation (7) has the estimate

$$\left|\bar{\tilde{K}}(t,\tau)\right| \le C\left\{t^{2\omega-2} + t^{\omega-1} + \frac{1}{\sqrt{t-\tau}} + \frac{t^{\omega-1}}{\sqrt{t-\tau}}\right\},$$

which means that the integral equation (2) for any

$$t^{\frac{3}{2}-\omega}\cdot f(t) \in L_\infty(0,\infty)$$

has a unique nonzero solution: $t^{\frac{3}{2}-\omega}\cdot\varphi(t) \in L_\infty(0,\infty)$.

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Псевдо-Вольтерраның интегралдық теңдеуінің шешуі туралы

Мақалада біртекті сингулярлы 2-текті Вольтерра интегралды теңдеуі (псевдо-Вольтерра интегралдық теңдеуі) қарастырылған. Оның ядросының қасиеттері дәлелденген. Характеристикалық теңдеуі құрастырылып, зерттеліп отырған интегралдық теңдеудің характеристикалық теңдеуі болатыны көрсетілді. Оның интегралдық операторының ядросының бағалауы анықталды. Сонымен қатар сәйкес біртекті емес интегралдық теңдеудің шешімі туралы сұрақтар қарастырылды, оның шешімінің жалғыздық класы анықталды. Сонымен бірге зерттеліп отырған біртекті емес теңдеудің оң жағы үшін салмақтық класы тағайындалды. Оның шешімінің жалғыздығының салмақтық класы теңдеудің интегралдық операторының бағалауы негізінде орнықтырылған.

Кілт сөздер: сипаттамалық теңдеу, ядро, интегралдық оператор, елеулі шенелген функциялар кластары.

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К решению одного псевдо-Вольтеррового интегрального уравнения

В статье изучено однородное сингулярное интегральное уравнение Вольтерра второго рода (псевдо-Вольтеррово интегральное уравнение). Доказаны свойства его ядра. Построено характеристическое уравнение. Показано, что оно действительно является характеристическим уравнением исследуемого интегрального уравнения. Установлены оценки ядра его интегрального оператора. Рассмотрены также вопросы разрешимости соответствующего неоднородного интегрального уравнения. Определен весовой класс единственности для его решения. Также установлен весовой класс для правой части исследуемого неоднородного уравнения. Весовой класс единственности его решения установлен на основе оценок ядра интегрального оператора уравнения.

Ключевые слова: характеристическое уравнение, ядро, интегральный оператор, класс существенно ограниченных функций.

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