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Nonlocal problem for degenerating elliptic-hyperbolic equations

In this paper investigated issues L_2-strong solvability of nonlocal problems for a degenerate equation of mixed elliptic-hyperbolic type. The peculiarity of this problem is the area in the hyperbolic part is not characteristic triangle, and the availability of additional internal boundary condition. Criterion of uniquely strong solvability of the problem is found.

Key words: boundary problem, degenerate equation of mixed type L_2-strong solvability.

Statement of the problem

We consider the following boundary value problem:

$$Lu = -tD_t^2u(x,t) - D_x^2u(x,t) = f(x,t),\ (t,x) \in Q; \qquad (1)$$

$$D_x^ju(0,t) = D_x^ju(2\pi,t),\ j = 0,1,\ t \in (-T,T); \qquad (2)$$

$$D_t^ju(x,-T) = \mu_j D_t^ju(x,T),\ j = 0,1,\ x \in (0,2\pi); \qquad (3)$$

$$u(x,t^*) = \psi(x),\ x \in (0,2\pi), \qquad (4)$$

where $D_t = \partial/\partial t,\ D_x = \partial/\partial x,\ Q = \{x,t \mid 0 < x < 2\pi, -T < t < T\}$.
Next, we put that

$$\begin{cases} t^{-\varepsilon}\cdot f(x,t) \in L_2(Q), \varepsilon > 0, \psi \in W_2^1(0,2\pi), \mu_j \in C, j = 0,1, \\ T < +\infty, \end{cases} \qquad (5)$$

$t^* \in [-T,0]$ — is the fixed point.

For problem (1)–(3) we shall study questions of existence of L_2-strong solution which, in addition to the conditions of continuity of the solution on the line $t = 0$ of the parabolic degeneracy must satisfy the condition (4).

Equation (1) at $t > 0$ belongs to the elliptical type, and at $t < 0$ — it belongs to the hyperbolic type. Boundary value problems for equations of mixed type (1) have been studied, for example, in [1–6].The problem (1)–(3) differs from the previously considered ones by that, firstly, in the hyperbolic part the domain is not characteristic triangle, secondly, by the presence of additional conditions (4). In [3] the Dirichlet problem for equation (1) was considered. It was possible to apply and develop the methods proposed in [4] for equations with П-operator coefficients to equations of mixed type (1). However, as follows from the obtained results, it can be developed for equations with coefficients that are not P-operators.

The main purpose of this article is to examine the issues of L_2-strong solvability of the boundary value problem (1)–(3) under the conditions (5).

Criterion for the unique strong solvability

We introduce the necessary hereinafter definition and notations.

Definition. Function $u(x,t) \in L_2(Q)$ is called L_2-strong solution of problem (1)–(4) if the sequence $\{u^{(n)}(x,t)\}_{n=1}^{\infty} \subset C_{x,t}^{2,2}(Q\setminus\{t=0\}) \cap C(Q)$, satisfying (2)–(4) will exist and such that

$$u^{(n)} \to u(x,t), t^{-\varepsilon}Lu^{(n)}(x,t) \to t^{-\varepsilon}f(x,t) \text{ при } n \to \infty \text{ в } L_2(Q)(\varepsilon > 0).$$

Let $J_v(z), N_v(z), I_v(z), K_v(z)$ — are the cylindrical functions (respectively, Bessel and Neumann functions and modified Bessel functions).

$$\eta_s(t,\tau)=\begin{cases}2\sqrt{t\tau}\left[I_1\left(2s\sqrt{t}\right)K_1\left(2s\sqrt{\tau}\right)-I_1\left(2s\sqrt{\tau}\right)K_1\left(2s\sqrt{t}\right)\right], 0<t\le\tau\le T;\\ \pi\sqrt{t\tau}\left[J_1\left(-2s\sqrt{-t}\right)N_1\left(-2s\sqrt{-\tau}\right)-J_1\left(-2s\sqrt{-\tau}\right)N_1\left(-2s\sqrt{-t}\right)\right], -T\le\tau\le t<0;\end{cases} \quad (6)$$

$$\Delta_s=\begin{pmatrix}J_0\left(-2s\sqrt{T}\right)-\mu_0 I_0\left(2s\sqrt{T}\right)s^{-1} & \sqrt{T}\left[J_1\left(-2s\sqrt{T}\right)-\mu_1 I_1\left(2s\sqrt{T}\right)\right]\\ -D_\tau^1\eta_s^2\left(t^*,\tau\right)_{|\tau=-T} & \eta_s^2(t^*,-T)\end{pmatrix}.$$

Here $\eta_s(t,\tau)$ is the Cauchy function [7] for the problems (13)–(14) $Y=\{s|\ s=0,\pm1,\pm2,...\}$:

Theorem. Boundary value problem (1)–(4) for any $t^{-\varepsilon}f\in\in L_2(Q)$, $(\varepsilon>0)$, $\psi\in W_2^1(0,2\pi)$ has L_2— unique strong solution if and only if the following conditions are satisfied:

$$\left|\Delta_s\right|\ne0\ \forall_s\in Y. \quad (7)$$

Here and further $|B|$ is determinant of the matrix B. Condition (7) in terms of data (5) gives a complete description of the correct boundary value problems of the form (1)–(4).We note that from this theorem the following assertions directly follow.

Assertion 1. If in the condition (4) $t^*=-T$, is accepted then the solvability conditions (7) of the problem (1)–(4) take the form:

$$s^{-1}\sqrt{T}\left[J_1\left(-2s\sqrt{T}\right)-\mu_1 I_1\left(2s\sqrt{T}\right)\right]\ne0\ \forall s\in Y. \quad (8)$$

Conditions (8) can be executed not for all s. For example, these conditions are violatedfor $s=0$ if $\mu_1=-1$.

Assertion 2. If in the condition (4) $t^*=0$, is accepted then the solvability conditions (7) of the problem (1)–(4) take the form:

$$\forall_s\in Y\left|\Delta_{s,\mu_1\mu_2}\right|=s^{-1}\sqrt{T}\left[\mu_0 J_1\left(-2s\sqrt{T}\right)I_0\left(2s\sqrt{T}\right)-\mu_1 J_0\left(-2s\sqrt{T}\right)I_1\left(2s\sqrt{T}\right)\right]\ne0. \quad (9)$$

In this case, at $\mu_0=\mu_1$ the boundary value problem (1)–(4) is uniquely L_2— strongly solvable in $L_2(Q)$, since the conditions:

$$s^{-1}\sqrt{T}\left[J_1\left(2s\sqrt{T}\right)I_0\left(2s\sqrt{T}\right)+J_0\left(2s\sqrt{T}\right)I_1\left(2s\sqrt{T}\right)\right]\ne0,$$

will be performed for all $s\in Y$ by reason of the linear independence functions $J_0\left(2s\sqrt{T}\right)$ and $I_0\left(2s\sqrt{T}\right)$, and also by reason of the validity of the following relations $J_0^{|}(z)=-J_1(z)$, $I_0^{/}(z)=I_1(z)$.

Proof of theorem. Decomposing right-hand side of equation (1):

$$f(x,t)=\sum_{s\in Y}f_s(t)\exp\{is\cdot x\},\ \{x,t\}\in Q, \quad (10)$$

we will seek the solution of the problem (1)–(4) in the form:

$$u(x,t)=\sum_{s\in Y}u_s(t)\exp\{is\cdot x\},\ \{x,t\}\in Q. \quad (11)$$

In this case, the boundary value problem (1)–(3) reduces to the study of boundary value problems for a countable system of ordinary differential equations:

$$\begin{cases}-tD_t^2u(t)+s^2\cdot u_s(t)=f_s(t), t\in(-T,T);\\ D_t^ju_s^1(T)=\mu_jD_t^ju_s^2(-T), j=0,1, s\in Y.\end{cases} \quad (12)$$

Introducing the auxiliary system of numbers $\{v_s, s\in Y\}$, $\{\varphi_s, s\in Y\}$, until temporarily unknown, instead of (12) we consider the following boundary value problems:

$$\begin{cases}-tD_t^2u_s(t)+s^2\cdot u_s(t)=f_s(t), t\in(0,T);\\ u_s(T)=\mu_o\varphi_s, D_t^1u_s(T)=\mu_1v_s, s\in Y;\end{cases} \quad (13)$$

$$\begin{cases}-tD_t^2u_s(t)+s^2\cdot u_s(t)=f_s(t), t\in(-T,0);\\ u_s(-T)=\varphi_s, D_t^1u_s(-T)=v_s, s\in Y.\end{cases} \quad (14)$$

For problems (13) and (14) we obtain the following representation for their solutions:

$$u_s(t)=\int_t^T \eta_s(t,\tau)\frac{f_s(\tau)}{\tau}d\tau+\varphi_s\mu_0\left[-D_\tau^1\eta_s(t,\tau)\right]_{|\tau=T}+v_s\mu_1\eta_s(t,T),\ t\in[0,T],s\in Y; \tag{15}$$

$$u_s(t)=-\int_{-T}^t \eta_s(t,\tau)\frac{f_s(\tau)}{\tau}d\tau+\varphi_s\left[-D_\tau^1\eta_s(t,\tau)\right]_{|\tau=-T}+v_s\eta_s(t,-T),\ t\in[-T,0],s\in Y; \tag{16}$$

where the functions $\eta_s(t,\tau)$ are functions of Cauchy [7] defined by (6).

Here the quantities $v_s,\varphi_s,s\in Y$ are yet unknown. To find these unknowns, we use, firstly, conditions of continuity of solution on the line $t=0,$ and secondly, the condition (4) by taking into account the following of expansion

$$\psi(x)=\sum_{s\in Y}\psi_s\exp\{isx\}. \tag{17}$$

As a result, for each $s\in Y$ we have:

$$\begin{cases}\left[J_0\left(-2s\sqrt{T}\right)-\mu_0 I_0\left(2s\sqrt{T}\right)\right]\varphi_s-s^{-1}\sqrt{T}\left[J_1\left(-2s\sqrt{T}\right)-\mu_1 I_1\left(2s\sqrt{T}\right)\right]v_s=F_s;\\ \left[-D_\tau^1\eta_s(t^*,\tau)\right]_{\tau=-T}\varphi_s+\eta_s(t^*,T)v_s=\psi_s,\end{cases} \tag{18}$$

where $F_s=s^{-1}\int_0^T\tau^{-0,5}\left[I_1\left(2s\sqrt{\tau}\right)f_s(\tau)+J_1\left(-2s\sqrt{\tau}\right)f_s(-\tau)\right]d\tau,\ \Psi_s=\psi_s+\int_{-T}^{t^*}\eta_s(t^*,\tau)\tau^{-1}f_s(\tau)d\tau.$

Condition for the solvability of system of algebraic equations (18) is its non-zero determinant, which defines the condition (7) of the theorem.

Now we define from the system of equations (18) unknown quantities φ_s,v_s by the formulas

$$\varphi_s=\frac{\left|\Delta_{\varphi_s}\right|}{\left|\Delta_s\right|},v_s=\frac{\Delta_{v_s}}{\Delta_s},\forall s\in Y, \tag{19}$$

where as usual, matrices $\Delta_{\varphi_s},\Delta_{v_s}$ are obtained from the matrix Δ_s by the replacement of the corresponding columns elements F_s,Ψ_s.

Next, substituting (19) into (15) and (16), we obtain the final representation of the solutions of boundary value problems (13)–(14):

$$u_s(t)=\int_t^T\eta_s(t,\tau)\tau^{-1}f_s(\tau)d\tau+\frac{\left|\Delta_{\varphi_s}\right|}{\left|\Delta_s\right|}\mu_0\left[-D_\tau^1\eta_s(t,\tau)\right]_{|\tau=T}+\frac{\left|\Delta_{v_s}\right|}{\left|\Delta_s\right|_s}\mu_1\eta_s(t,T),\ t\in(0,1),s\in Y; \tag{20}$$

$$u_s(t)=-\int_{-T}^t\eta_s(t,\tau)\tau^{-1}f_s(\tau)d\tau+\frac{\left|\Delta_{\varphi_s}\right|}{\left|\Delta_s\right|}\left[-D_\tau^1\eta_s(t,\tau)\right]_{|\tau=-T}+\frac{\left|\Delta_{v_s}\right|}{\left|\Delta_s\right|}\eta_s(t,-T),\ t\in(-T,0),s\in Y; \tag{21}$$

Now we get L_2— estimates for the solutions (20)–(21) that are uniform at $s\in Y,$ i.e. estimates of the form (here $\varepsilon>0$):

$$\left\|u_s(t)\right\|_{L_2(0,T)}\le C_1\left[\left\|t^{-\varepsilon}f_s(t)\right\|_{L_2(0,T)}+\left|s\psi_s\right|\right]\ \forall s\in Y; \tag{22}$$

$$\left\|u_s(t)\right\|_{L_2(-T,0)}\le C_2\left[\left\|t^{-\varepsilon}f_s(t)\right\|_{L_2(-T,0)}+\left|s\psi_s\right|\right]\ \forall s\in Y; \tag{23}$$

where the constants C_1,C_2 do not depend on s.

First, we'll consider the estimates (22) of solutions (20). We estimate each summand from (20) separately. For the integral summand, we will have:

$$\left|\int_t^T\eta_s(t,\tau)\tau^{-1}f_s(\tau)d\tau\right|^2\le T\left\|\tau^{\varepsilon-1}\eta_s(t,\tau)\right\|_{L_2((0,T)\times(0,T))}\cdot\left\|\tau^{-\varepsilon}f_s(\tau)\right\|^2_{L_2(0,T)}. \tag{24}$$

According to (6) for finite s function $\tau^{\varepsilon-1}\eta_s(t,\tau)$ on the right of (24) has a singularity only at $t\to0+$ and $\tau>0.$ Namely, expression $\tau^{\varepsilon-1}\eta_s(t,\tau)$ has the order $\tau^{\varepsilon-0,5},$ i.e. has a squared integrable singularity. It remains to consider the case, when $|s|\to\infty.$ In this case, using the asymptotic representation of cylindrical functions $I_1(z)$ and $K_1(z)$ for large values of the argument, for the above multiplier we obtain the order, equal to s^{-1} that shows the boundedness of this multiplier in this case under consideration.As a result, we obtain the boundedness of the first multiplier from (24) by constant, independent of s.

We estimate the second summand in (20). As well as in the estimate of the first summand, we will analyze the case: $t \to 0+$ and $|s| \to \infty$. At $t \to 0+$ and at finite s expression $\left[-D_\tau^1 \eta_s(t,\tau)\right]_{|\tau=T}$ tends to $I_0(2s\sqrt{T})$, and in the case $|s| \to \infty$ it tends to a finite value. As for large s determinant $\left|\Delta_{\varphi_s}\right|$ has the order $|s|^{-2,5} \exp\left\{2s\sqrt{T}\right\}$, and $|\Delta_s|$ has $|s|^{-1,5} \exp\left\{2s\sqrt{T}\right\}$, respectively, then their ratio $\left|\Delta_{\varphi_s}\right| / |\Delta_s|$ has the order s^{-1} .Thus, the second summand in (20) is also bounded by a constant independent of s.

It remains to estimate the third summandin (20). In this case we have

$$\eta_s(t,T) \to s^{-1}\sqrt{T} I_1(2s\sqrt{T}) \text{ at } t \to 0.$$

And in the case when $t > 0, |s| \to \infty$, it has the order s^{-1}. By virtue of the fact that Δ_{v_s} has the order $|s|^{-1,5} \exp\left\{2s\sqrt{T}\right\}$, the ratio $\left|\Delta_{v_s}\right| / |\Delta_s|$ is bounded. Thus, the boundedness of the third summandin (20) by a constant independent of s follows from here.

So we have established the validity ofthe estimate (22).

Using the properties of cylindrical functions $J_n(z)$ and $N_n(z)$ at $|z| \to 0$ and $|z| \to \infty$, analogously the validity of a priori estimate (23) for the solution (21) is established.

Finally, fromestimates (22) and (23) a priori estimate for the solution of (12) directly follows:

$$\|u_s(t)\|_{L_2(-T,T)} \le C\left[\left\|t^{-\varepsilon} f_s(t)\right\|_{L_2(-T,T)} + \left|s\psi_s\right|\right], \forall_s \in Y, \qquad (25)$$

where constant C independent of s.
Now, using the equality

$$\|u(x,t)\|^2_{L_2(Q)} = 2\pi \sum_{s \in Y} \|u_s(t)\|^2_{L_2(-T,T)},$$

From (25) a priori estimate for the solution of the boundary value problem (1)–(4) follows:

$$\|u\|_{L_2(Q)} \le C\left\|t^{-\varepsilon} Lu\right\|_{L_2(Q)}, \qquad (26)$$

where operator $t^{-\varepsilon}L$ is defined as the closure of the differential operation L in $L_2(Q)$ from (1), defined on smooth functions subject to the boundary conditions (2)–(3) and condition (4) (that corresponds to the definition for L_2— strong solution). Thus, the estimate (26) is valid for any function from definition domain of operator $t^{-\varepsilon}L$, i.e. the bounded inverse operator exists:

$$L^{-1}t^{\varepsilon} \text{ c } D(L^{-1}t^{\varepsilon}) \equiv L_2(Q).$$

Hence, the boundary value problem (1)–(4) is uniquely solvable for arbitrary $t^{-\varepsilon} f \in L_2(Q), \psi \in W_2^1(0,1)$. Indeed, $D(L^{-1}t^{\varepsilon})$ contains all finite sums of the form (10), multiplied by $t^{-\varepsilon}$, and all finite sums of the form (17), therefore, the operator $D(L^{-1}t^{\varepsilon})$ is given on a dense set and by virtue of the boundedness it can be extended by continuity to the entire space of functions for which $t^{-\varepsilon} f \in L_2(Q), \psi \in W_2^1(0,1)$.

Thus, at the unique solvability of boundary value problems (12) and at the presence of estimates (25), the validity of the theorem follows.

On the other hand, at disarrangement of the unique solvability of at least one of the boundary value problems (12), we have that a nontrivial solution $u_s(t)$ of the corresponding homogeneous equation exists and, in this case, the function $u_s(t) \cdot \exp\{isx\}$ is a non-trivial solution of the homogeneous problem (1)–(4).

In this case when at the unique solvability of each boundary value problems (12) the presence of a priori estimates (25) that are uniform on s is violated, i.e. there are sequences $\left\{f_{s^n}\right\}_{n=1}^{\infty}$, $\left\{\psi_{s^n}\right\}_{n=1}^{\infty}$ such that

$$\left\|u_{s^n}(t)\right\|_{L_2(-T,T)} \ge C\left[\left\|t^{-\varepsilon} f_{s^n}(t)\right\|_{L_2(-T,T)} + \left|s\psi_s\right|\right], \; n = 1, 2, ...,$$

then the operator $L^{-1}t^{\varepsilon}$ is bounded on a dense set (on the finite sums of the type (10) multiplied by $t^{-\varepsilon}$, and on the finite sums of the type (17)). However, it is unbounded. This gives a proof of the necessity of the theorem conditions.

Thus, theorem is completely proved.

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Өзгешеленетін эллиптика-гиперболалық теңдеуі үшін локалды емес есеп

Мақалада эллиптика-гиперболалық типтес аралас өзгешеленетін теңдеуі үшін локалды емес есебінің L_2-күшті шешімділігі зерттелді. Берілген есептің ерекшелігі гиперболалық бөлігіндегі облыстың сипаттамалық үшбұрыш болмауында және де қосымша ішкі шеттік шарттын бар болуында. Қарастырылған есептің бірмәнді күшті шешімділігінің критерийі табылған.

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Нелокальная задача для вырождающегося эллиптико-гиперболического уравнения

В статье исследованы вопросы L_2-сильной разрешимости нелокальной задачи для вырождающегося уравнения смешанного эллиптико-гиперболического типа. Особенностью данной задачи является то, что область в гиперболической части не является характеристическим треугольником, а также наличие дополнительного внутреннекраевого условия. Найден критерий однозначной сильной разрешимости рассматриваемой задачи.

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