



Positive Jonsson Theories

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This paper is dedicated to the memory of Tölendi Garifuliy.

Abstract. This paper is a general introduction to Positive Logic, where only what we call h-inductive sentences are under consideration, allowing the extension to homomorphisms of model-theoretic notions which are classically associated to embeddings; in particular, the existentially closed models, that were primitively defined by Abraham Robinson, become here positively closed models. It accounts for recent results in this domain, and is oriented towards the positivisation of Jonsson theories.

Résumé. Cet article est une introduction générale à la Logique Positive, où seuls sont considérés les énoncés dits h-inductifs, ce qui permet d'étendre aux homomorphismes les notions de Théorie des Modèles classiquement associées aux plongements; en particulier les modèles existentiellement clos, primitivement définis par Abraham Robinson, deviennent ici les modèles positivement clos. Il tient compte de résultats récents en ce domaine, et se focalise sur ce que deviennent les théories de Jonsson dans un contexte positif.

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1. Why Positive Logic?

A tradition was established in the fifties of dividing the Theory of Models for First Order Logic into two schools.

On the west coast of the USA, Alfred Tarski and his faithful disciple Robert Vaught considered arbitrary sentences and arbitrary structures, but restricted the morphisms between them to elementary embeddings, that is, embeddings respecting the truth value of each formula. The modern output of this school is the monster model of a complete theory, the Liebenraum of a great majority of model theorists.

On the east coast, on the contrary, Abraham Robinson put no restrictions on embeddings, but considered only inductive sentences and a special kind of structures, the existentially closed models of a given inductive theory; embeddings between such models preserve the satisfaction of existential formulae in both directions (for a general reference, see Robinson [29]).

It was fully realized only at the beginning of this millenium that this Robinson's logic could be extended one step further: to-day, it appears to be a special case of the Theory of Models for Positive Logic, where general homomorphisms are considered, and not only embeddings. In Positive Logic, we consider only positive existential formulae, and a special kind of inductive sentences that we call h-inductive (for the reason that they are preserved under inductive limits of homomorphisms, and not only under inductive limits of embeddings, as are the inductive sentences of Robinson). More precisely, Robinson's setting corresponds to the special positive case where the language is expanded by relation symbols interpreting the negations of the atomic formulae, transforming these negations into positive beings (the fact that two positive formulae are complementary is expressed by h-inductive axioms).

In fact, the tools used in Positive Logic were known long time ago (Chang and Keisler [9, ex. 5.2.24], Makkai and Reyes [17] and the categorical model-theorists of the Province de Québec in the seventies), and so was the process commonly called Morleyisation, by which an incrementation of the language allows the interpretation of Tarski's logic into Robinson's logic (see Gödel [12]!), and Robinson's logic into Positive Logic. But what was understood by Itai Ben Yaacov [2,3], etc.) is that:

- (i) the universal domains obtained in Robinson's context are more general than the ones of Tarski, and the universal domains obtained in Positive Logic are more general than the ones of Robinson,

- (ii) even when we work in a Tarskian frame, it is sometimes necessary to abandon the negation for a fine description of the structures under study,
- (iii) Positive Logic has maximal generality if compactness is preserved; in other words, Ben Yaacov was able to reconstruct the universal domains from their spaces of types, in a way reminding the reconstruction of the models from the algebra of formulae by the fathers of Algebraic Logic (Halmos, Henkin, Monk, Daigneault; see Daigneault [10]).

Under the present view of Positive Logic, Robinson's restriction appears to be highly inadequate, since on one hand all the results obtained by himself and his followers extend in a quite straightforward manner to the general positive case, and on the other this general positive case is truly wider, allowing practical applications that Robinson's frame does not permit; for instance, one of the original motivations of Ben-Yaacov was the study of quotients by infinitely definable equivalence relations.

Indeed, a typical feature of Positive Model Theory is that it does not distinguish substantially between definable and infinitely definable sets, since it is innocuous for it to expand the language by the introduction of a new relation symbol denoting an infinite conjunction of positive formulae, provided that one is only interested in the structures that are sufficiently positively saturated. This cannot be done in Robinson's frame, which would force us to introduce also the negation of this new atomic formula, affecting drastically the model-theoretic properties of the structures under consideration.

Certainly, Abraham Robinson had everything at his disposal to develop his theory in the general positive frame; but he did not, so forcing us to rewrite many of his results with a positive tag. This is an easy, but not a vain, exercise; an immediate reward is that the positive proofs go smoother: Positive Logic is very direct, as it allows more freedom in the literal sense (for instance, free amalgams exist for homomorphisms, not for embeddings); examples and counter-examples are easy to find in Positive Logic, and experience shows that, after some elaboration, they can be transformed into examples and counter-examples valid in the robinsonian setting. In some sense, Positive Logic throws some clarifying light on Robinson's theory itself.

The fact is that we are embarrassed, whenever we extend to the general positive context some notion that was originally defined by Robinson, to be compelled to mark it with prefixes like *pos.* (for positive) or *h-* (for homomorphism), since our intimate belief is that it is the robinsonian special notion that should be marked; sometimes we drop the positive tag, hoping that no confusion will arise.

The present paper provides an elementary exposition to the theory of models for Positive Logic, and contains the extension of many results previously obtained by Robinson's disciples, in particular a positivisation of Jonsson theories (for a survey of Jonsson theories in the robinsonian context, see Yeshkeev [31]); in so doing, we revise more than sixty years of activity in Model Theory. It describes situations that cannot occur in a negative context (we admit that, when first seen, these new phenomena may have a certain power of disturbance on the minds of the logicians which have been educated under

Robinson's portico), and also accounts for some very recent works in Positive Logic. The proofs are most of the time sketchy, or omitted when we can provide a reference (mainly from the detailed exposition paper Ben Yaacov and Poizat [7]); in a few cases we quote arguments from published works, when we think that they are essential for a reasonable self-countenance of our paper.

We shall not adopt here a strict yaacobian ideology; that is, we shall not focus our attention only on the positively saturated models of our theories, nor attempt to give an account of forking in this context (Pillay [25] in a robinsonian frame; Ben Yaacov [4–6], etc.). We shall simply, possibly in a naive way, consider the h-inductives theories and their models as objects of study per se, and even describe some elementary classes or properties, in the language of the full First Order Logic which negation, which are associated to them.

2. Basic Definitions

2.1. Homomorphisms and Positive Formulae

We consider \mathcal{L} -structures in a fixed, but arbitrary, language \mathcal{L} involving individual constants, functions and relations; it is convenient to allow the presence of 0-ary relation symbols, that is, of propositional constants (they are necessary for the Morleyisation process used in the proof of the Compactness Theorem in Sect. 2.3). The language always contains a binary relation symbol $=$ denoting equality, and a 0-ary symbol \perp denoting antilogy (this is the only systematic way to provide a positive definition to the empty set).

To avoid useless complications, we adopt the usual convention that the underlying set of a structure should be non-empty; therefore we are dispensed to introduce a specific symbol for tautology, positively defined as $(\exists x)x=x$.

By definition, an *homomorphism* between two \mathcal{L} -structures M and N is a map h from M to N such that, for every individual constant c , every function symbol f and every relation symbol r of \mathcal{L} , and every tuple \underline{a} of elements of M , the following holds:

- $h(c_M) = c_N$
- $h(f_M(a_1, \dots, a_m)) = f_N(h(a_1), \dots, h(a_m))$
- if $M \models r_M(a_1, \dots, a_n)$ then $N \models r_N(h(a_1), \dots, h(a_n))$.

In other words, any atomic formula satisfied by \underline{a} in M is satisfied by $h(\underline{a})$ in N .

When there is an homomorphism from M to N , we say that N is a *continuation* of M . We observe that a continuation of M is nothing but a model of the *positive diagram* $\text{Diag}^+(M)$ of M , which is the set of atomic sentences satisfied by M in the language $\mathcal{L}(M)$ obtained by adding to \mathcal{L} individual constants naming the elements of M .

We say that the homomorphism h is an *embedding* if moreover h is injective and, for every tuple \underline{a} in M and every relation symbol r in the language, we have: $M \models r_M(a_1, \dots, a_n)$ if and only if $N \models r_N(h(a_1), \dots, h(a_n))$; in other words, an atomic formula is satisfied by \underline{a} in M if and only if it is satisfied by $h(\underline{a})$ in

N . An *isomorphism* is a bijective embedding. If there is an embedding from M to N , that is, if M is isomorphic to a substructure of N , we say as usual that N is an *extension* of M .

By definition, a *positive formula* is obtained from the atomic formulae by the use of \vee , \wedge and \exists (Caveat: no universal quantifiers). It can be written in prenex form as $(\exists \underline{x}) \varphi(\underline{x})$, where φ is positive quantifier-free; φ in turn can be written as a finite disjunction of finite conjunctions of atomic formulae.

An immediate, but fundamental, observation is the following: *If h is an homomorphism from M to N and \underline{a} is a tuple of elements of M , then every positive formula satisfied by \underline{a} in M is satisfied by $h(\underline{a})$ in N .* It can be proved by an easy induction on the complexity of the formula, or by the consideration of its prenex form.

If every tuple \underline{a} in M satisfies the same positive formulae as its image $h(\underline{a})$ in N , we say that h is a *pure* homomorphism, or an *immersion*. An immersion is in particular an embedding.

Isomorphisms, and more generally elementary embeddings for the full First Order Logic with negation, are obvious immersions; other examples are the *retractile* homomorphisms h from M to N , for which we can find an homomorphism g from N to M such that $g \circ h$ is an automorphism of M .

We say that M is *positively closed* (in short, pc) inside a class Γ of structures if every homomorphism from M to any N in Γ is pure.¹

We say that M is *diagram-maximal* (in short, dm) inside Γ if every homomorphism h from M to any N in Γ is an embedding.²

If the class Γ contains the *Terminus Structure*, formed by a single element satisfying every atomic formulae except the antilogy, then Terminus is the only pc, and even dm, element of Γ . We observe that Terminus satisfies all the sentences of the form $(\forall \underline{x})(\exists \underline{y}) \varphi(\underline{x}, \underline{y})$, where φ is positive quantifier-free and free from \perp .

Dually, the class Γ may contain an *initial* structure, which is a structure that can be continued in a unique way into any member Γ ; when it exists, this initial model is unique up to isomorphism. For instance, when Γ is the class of groups, the trivial group Terminus is also the initial group.

A useful fact is that, provided that the language \mathcal{L} contains at least one individual constant, any set Ta of atomic sentences not containing \perp has an initial model; we leave to our readers its proof, which is quite straightforward when no function is involved.

2.2. H-Inductive and h-Universal Theories

An *h-inductive* sentence is by definition equivalent to a finite conjunction of sentences each of them declaring that a certain positively defined set is included into another. Such a simple h-inductive sentence has the form $(\forall \underline{x}) [(\exists \underline{y}) \varphi(\underline{x}, \underline{y})$

¹ Positively closed structures were called *positivement existentiellement closes* in Ben Yaacov and Poizat [7], and pc in some other papers.

² Dm structures have been introduced under the name of *h-maximal* structures in Kungozhin [16]; we prefer to name them diagram-maximal because their positive diagram is maximal for a structure continuable in a member of Γ .

$\Rightarrow (\exists z) \psi(\underline{x}, z]$, and its prenex form is of the kind $(\forall \underline{u})(\exists \underline{v}) \neg \varphi'(\underline{u}) \vee \psi'(\underline{u}, \underline{v})$, where φ , φ' , ψ and ψ' are positive quantifier-free; note that the existential quantifier spans only the positive part of the disjunction.

It is easily seen that the disjunction or the conjunction of two h-inductive sentences is also h-inductive; but the conjunction of two simple h-inductive sentences may not be equivalent to a simple one (to find a counter-example is a good exercise in Boolean Calculus, that we also leave to our readers).

In Positive Logic only h-inductive sentences are under consideration.

When $(\exists \underline{y}) \varphi(\underline{x}, \underline{y})$ is the tautology $(\exists t) t = t$, we obtain sentences of the kind $(\forall \underline{u})(\exists \underline{v}) \psi(\underline{u}, \underline{v})$, where $\psi(\underline{u}, \underline{v})$ is positive quantifier-free, which are therefore h-inductive; we shall call them *positive inductive*, in spite of the fact that they are not always positive formulae in our sense (when the universal quantifier is present); this kind of sentences declares that everybody satisfies some positive condition. In particular, sentences like $(\forall \underline{u}) \psi(\underline{u})$ or $(\exists \underline{v}) \psi(\underline{v})$, where ψ is positive quantifier-free, are h-inductive. We are in the necessity to introduce other sentences than the positive inductive ones, with a somehow negative content, if we wish to reach a final destination other than Terminus.

When the positive part $\psi'(\underline{u}, \underline{v})$ of the disjunction is absent, the sentence has the form $(\forall \underline{u}) \varphi'(\underline{u}) \Rightarrow \perp$, and is a special case of an h-inductive sentence, which we call *h-universal*; it can be written $(\forall \underline{x}) [\neg (\exists \underline{y}) \varphi(\underline{x}, \underline{y})]$, or otherwise $(\forall \underline{u}) \neg \varphi'(\underline{u})$, that is $\neg (\exists \underline{u}) \varphi'(\underline{u})$. An h-universal sentence declares that a certain positively defined set is empty. It would be unwise to call such a sentence positively universal, since it is precisely the negation of a positive sentence. In particular, the negation of an atomic sentence is h-universal.

The conjunction or the disjunction of two h-universal sentences is equivalent to an h-universal sentence.

If an h-universal sentence is satisfied in a continuation of M, then it is also true in M. One can see that this property characterizes the h-universal sentences among the sentences of the full First Order logic with negation (Ben Yaacov and Poizat [7], Lemme 21); for a positivist fanatic this converse is not truly important, and in fact we shall not use it.

We also observe that, if M is immersed in N, then any h-inductive sentence satisfied by N is also satisfied by M.

It is easily seen, writing its boolean part in conjunctive form, that every *universal* sentence $(\forall \underline{x}) \nu(\underline{x})$, where ν is quantifier-free but possibly involves the negation, is h-inductive (but not always h-universal). By contrast, $(\exists \underline{x}) r_1(\underline{x}) \wedge \neg r_2(\underline{x})$ and $(\exists \underline{x})(\forall \underline{y}) r(\underline{x}, \underline{y})$ are not h-inductive.

A property, and in fact a characteristic property among the sentences of full First Order Logic (see Chang and Keisler [9, ex. 5.2.24], Ben Yaacov and Poizat [7, Théorème 23]), of the h-inductive sentences is that they are preserved under inductive limits of *homomorphisms*; once you know what these limits are, it becomes obvious that the two counter-examples above have not this preservation property.

We shall use inductive limits only to establish the equivalence of the Axiom of Choice to the following **Continuation Principle**: *Every model of an h-inductive theory T can be continued into a pc model of this theory.*

Since the truth of an axiom, by definition, cannot be proved, and since the Axiom of Choice is admitted in any domain of Modern Mathematics where it does not produce undesirable disturbances (and Model Theory is such a domain), our reader has the comfortable possibility to admit the Continuation Principle and skip the next section, unless he wishes to check the equivalence of this Principle to a better known formulation of Zermelo’s Axiom of Choice.

We conclude the present section by the description of two kinds of remarkable h-inductive sentences.

The first kind expresses that an $(n+1)$ -ary positive formula $\varphi(\underline{x},y)$ is the graph of an n -ary function: $(\forall \underline{x},y,z) \varphi(\underline{x},y) \wedge \varphi(\underline{x},z) \Rightarrow y = z$ together with $(\forall \underline{x})(\exists y) \varphi(\underline{x},y)$. Therefore there is no loss of generality if we assume that the language contains no functions, since the substitution of a function by its graph translates a positive formula into a positive one, and vice versa.

The second kind expresses that two positive formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ are each the negation of the other: $\neg(\exists \underline{x}) \varphi(\underline{x}) \wedge \psi(\underline{x})$ together with $(\forall \underline{x}) \varphi(\underline{x}) \vee \psi(\underline{x})$; note that the second sentence is *not* h-universal even when $\varphi(\underline{x})$ and $\psi(\underline{x})$ are quantifier-free. Such sentences permit to transform, by a mere expansion of the language, any inductive theory T in Robinson’s sense³ into an h-inductive theory: consider the language \mathcal{L}' obtained by adding for each relation symbol r of \mathcal{L} (including the equality symbol $=$) a new relation symbol r' , and the h-inductive theory T' formed by the axioms saying that r' interprets the complement of r , plus the axioms obtained by replacing in the axioms of T the negations of the atomic subformulae by their new positive expressions. T and T' have essentially the same models, since any model of T can be expanded in a unique way to a model of T' , and reciprocally any model of T' , when considered as an \mathcal{L} -structure, is a model of T . Homomorphisms between models of T' correspond to embeddings between models of T , and the pc models of T' corresponds to the ec (existentially closed) models of T .

2.3. Inductive Limits and Compactness

Considering an ascending sequence of \mathcal{L} -structures in an arbitrary language \mathcal{L} , each of them being a restriction of the next:

$$M_0 \subseteq M_1 \subseteq \dots \subseteq M_n \subseteq M_{n+1} \subseteq \dots$$

we can define the *inductive limit* M of the sequence as their common extension to the union of their underlying sets: an atomic formula is satisfied by a tuple of elements of M when it is satisfied in M_n for n large enough; this construction is so common in Model Theory that there is hardly a name for it.

When we have a sequence of embeddings:

$$M_0 \xrightarrow{e_0} M_1 \xrightarrow{e_1} \dots \xrightarrow{e_{n-1}} M_n \xrightarrow{e_n} M_{n+1} \xrightarrow{e_{n+1}} \dots$$

we can assimilate $e_n(M_n)$ to a substructure of M_{n+1} and define the inductive limit in the same way. But the limit exists also in the case of a sequence of homomorphisms, possibly non injective:

³ A theory axiomatized by sentences of the form $(\forall \underline{x})(\exists y) v(\underline{x},y)$, where v is quantifier-free, but possibly uses negation.

$$M_0 \xrightarrow{h_0} M_1 \xrightarrow{h_1} \dots \xrightarrow{h_{n-1}} M_n \xrightarrow{h_n} M_{n+1} \xrightarrow{h_{n+1}} \dots$$

The best way to define it is to add to the language \mathcal{L} individual constants naming the elements of the M_n , in a disjoint manner; the inductive limit is the initial model of the theory formed by the positive diagrams of the M_n and the equalities $c = f_n(c)$. One can say that this limit is the least common continuation of the M_n .

Since an atomic formula (in particular an equality) is true in the limit M provided that it is true finally in the M_n , then if all the M_n satisfy a certain h -inductive sentence, so does the limit.

A regrettable complication is that it is not enough to consider limits of countable sequences of homomorphisms: our reader will define, in a similar manner, the inductive limit of an arbitrary sequence M_i indexed by a totally ordered set I , with an homomorphism h_{ji} from M_i into M_j whenever $i < j$, such that $h_{kj} \circ h_{ji} = h_{ki}$ if $i < j < k$.

By definition, an *h-inductive* class is a class of \mathcal{L} -structure closed under inductive limits of homomorphisms. The class of models of any h -inductive theory is h -inductive (reciprocally, the theory of an h -inductive elementary class, in the sense of full First Order Logic, is axiomatized by h -inductive sentences; but this is not the case for the theory of an arbitrary h -inductive class: see Example 5 in Sect. 3.7). And also:

Lemma 1. *If T is an h -inductive theory, its pc models, and also its dm models, form h -inductive classes.*

Proof. If a tuple \underline{a} in the limit satisfies a positive formula $\varphi(\underline{a})$, there is in some M_i a tuple $\underline{\alpha}$ which projects on \underline{a} and satisfies $\varphi(\underline{\alpha})$ in M_i . □

Using her⁴ favourite variant of the Axiom of Choice, the reader will show that: *In an h -inductive class, every point can be continued into a pc element.* In particular, pc models exists for any consistent h -inductive theory.

In Ben Yaacov and Poizat [7], an example was given for the converse, that is, the implication from the Continuation Principle towards the Axiom of Choice. We alter it so that it functions in Robinson’s setting as well.

Example 1. Consider a non-empty set A and an equivalence relation E between the elements of A ; the language \mathcal{L} contains a predicate⁵ $r_a(x)$ for each a in A ; the h -universal theory T consists in the axioms $\neg (\exists x,y) r_a(x) \wedge r_b(y)$, for each pair (a,b) of distinct elements of A which are congruent modulo E ; T has a pc model (with one point !), or a dm model, or an ec model, if and only if A has a choice-subset for E . The variant T' of T , obtained by adding to it the axioms $\neg (\exists x) r_a(x) \wedge r_b(x)$ whenever $a \neq b$, does not need a set-theoric hypothesis to have dm models.

We now summarize the straightforward proof of the Compactness Theorem for full First Order Logic, based on inductive limits, that is offered in

⁴ Non-sexist languages like Qazaq, or even French in this case, would use here the same possessive for both genders.

⁵ We recall that, by definition, a predicate is a unary relation symbol.

Ben Yaacov and Poizat [7]. Since functions can be replaced by their graph, it is harmless to assume that the language contains only relations and individual constants, a thing which simplifies greatly the description of the initial models; we can assume also that at least one individual constant is present in the language, since by definition structures are non-void. Then the proof proceeds in three steps.

The first (Lemme 2) is quite obvious: if every finite subset of a theory $Ta \cup Tu$, where Ta is composed of atomic sentences and Tu is composed of h-universal sentences, has a model, then the initial model of Ta is a model of Tu .

The second (Lemme 3) consists in proving that, if T is an h-inductive theory, and if Tu is the set of h-universal sentences which are consequence of a finite subset of T , then any pc model of Tu is a model of T . Of course, when the Compactness Theorem is known, Tu is simply the set of h-universal consequences of T .

When T is finitely consistent,⁶ Tu is also finitely consistent, and consistent thanks to the first step; at this stage the Compactness Theorem for h-inductive theories is obtained by an application of the Continuation Principle to Tu (this is the only place where the Axiom of Choice is used in the proof).

What remains is to interpret the full First Order Logic into Positive Logic by an expansion of the language called *Positive Morleyisation*. It consists in the introduction of a new relation symbol r_φ for each formula φ of the logic with negation; provided that we do not write directly universal quantifiers in our formulae, but discompose them as $\neg\exists\neg$, the conditions compelling r_φ to be interpreted by φ are h-inductive, so that to any theory T of the full First Order Logic is associated an h-inductive theory T' in the expanded language, which has practically the same models as T ; T' is consistent if and only if T is consistent, and T' is finitely consistent if and only if T is finitely consistent, so that the proof is completed.

The only thing that is altered by the morleyisation process is the notion of homomorphism: the homomorphisms between the models of T' corresponds exactly to the elementary embeddings between the models of T .

Morleyisation is named after the paper Morley and Vaught [19], where this process is applied. In fact, this kind of expansion of the language belongs to the prehistory of the Theory of Models; for instance, in the famous paper of Kurt Gödel [12], it is taken for granted in the proof of the Completeness Theorem as well as in the proof of the Compactness Theorem that, as far as validity is concerned, it can be assumed that the sentences are inductive.

Positive Morleyisation was also well-known to the group of model-theorists of Montreal in the seventies (see Makkai and Reyes [17]); they were studying the h-inductive sentences under the name of *coherent sentences*.

We conclude the section by the following characterization of the pc models, which is a direct application of the Compactness Theorem:

⁶ By “finitely consistent” we mean that every finite fragment of T has a model.

Lemma 2. *In a pc model M of an h -inductive theory T , if a tuple \underline{a} of elements does not satisfy some positive formula $\varphi(\underline{x})$, it is because it satisfies another positive formula $\psi(\underline{x})$ which is contradictory to it in the sense of T : T implies the h -universal sentence $\neg(\exists \underline{x}) \varphi(\underline{x}) \wedge \psi(\underline{x})$.*

Proof. The theory formed by $\varphi(\underline{a})$, the positive diagram of M and T is contradictory. \square

In the other direction, a model having this property is obviously pc.

2.4. Positive Saturation

We say that a structure M is *positively ω -saturated* if, for every tuple \underline{a} in M , every set $\Phi = \{\dots \varphi_i(x, \underline{a}), \dots\}$ of positive formulae which is finitely satisfiable in M is realized in M . Note that, in this definition, we can replace the variable x by a fixed tuple \underline{x} of variables of an arbitrary length.

Lemma 3. (i) *Any pc model of an h -inductive theory T can be continued into a positively ω -saturated pc model of T .*
(ii) *Any model of an h -inductive theory T can be embedded, and even immersed, into a positively ω -saturated one.*

Proof. (i) Consider a pc model M_0 of T , and enumerate the sets of positive formulae $\Phi_0, \dots, \Phi_\lambda, \dots$ in one variable, with parameters in M_0 , which are finitely satisfiable in M_0 ; if we assign distinct variables to the Φ_λ , their union is consistent with $\text{Diag}^+(M) \cup T$, so that each of them is realized in some continuation M_1 of M_0 which is a model of T , that we may take pc. We apply the same process to M_1 that we continue in a model M_2 , and repeat. The inductive limit of the M_n is a pc model of T ; it is positively ω -saturated, because, since M_n is pc, every set of positive formulae with parameters in M_n , which is consistent with $\text{Diag}^+(M_n) \cup T$, is finitely satisfiable in M_n .

(ii) Given a structure M , the h -inductive sentences with parameters in M satisfied in M form an h -inductive theory $T(M)$, in the language $\mathcal{L}(M)$ where names for the elements of M are added to \mathcal{L} ; consider a positive formula $\varphi(\underline{a}, \underline{x})$ with \underline{a} in M : there is some \underline{b} in M satisfying $\varphi(\underline{a}, \underline{b})$ unless $\neg(\exists \underline{x}) \varphi(\underline{a}, \underline{x})$ belongs to $T(M)$, so that M is obviously a pc model of $T(M)$. We then apply the result above. \square

We can also define positive κ -saturation for any infinite cardinal κ , and prove a similar result of existence. We have the usual limitations of cardinality: typically, for any κ bigger than the size of the language, we obtain κ^+ -saturated models of cardinal at most 2^κ , or more generally κ -saturated models of cardinal at most λ when $\lambda^{<\kappa} = \lambda$, or a κ -saturated model of cardinal at most κ (yes; see Sect. 3.7) κ when κ is strongly inaccessible.

Note. By Lemma 2, two tuples extracted from a positively ω -saturated pc model of T which satisfy the same positive formulae are in an infinite back-and-forth correspondance, and therefore satisfy the same formulae of full First

Order Logic with negation. For the same reason, any embedding between such models is elementary.

3. Universal Domains

3.1. Companion Theories

We say that two h-inductive theories T and T' , in a same language \mathcal{L} , are *companion* if every model of one of them can be continued into a model of the other.

Therefore, for any model M of T , the theory formed by T' and $\text{Diag}^+(M)$ is consistent; by compactness, this means that every h-universal consequence of T' is a consequence of T . By symmetry, we see that T and T' are companion if and only if they have the same h-universal consequences.

Moreover, T and T' are companion if and only if they have the same pc models. Indeed, if this is true, they are companion since any model of T can be continued into a pc model of T . Reciprocally, if T_u is the set of h-universal consequences of T , any pc model of T_u can be immersed into a model of T , and therefore is itself a model of T ,⁷ and obviously a pc one; moreover, since any model of T_u can be continued into a model of T , any pc model of T is also pc for T_u . In other words, T and T_u have the same pc models, and so do T and T' if they are companion.

Therefore, an h-inductive theory T has a minimal⁸ companion, which is T_u , and a maximal (h-inductive) one, which is the h-inductive theory T_k of its pc models, that is, the set of h-inductive sentences which are true in each of its pc models (T_k is not necessarily the full First Order theory of the pc models of T : see Example 5 in Sect. 3.7); T_k is indeed a companion of T since any model of T can be continued into a pc model; it is called, after Kaiser [15], the h-inductive *Kaiser hull* of T . Any h-inductive theory between T_u and T_k is a companion of T ; for instance, the theory T_m of the dm models of T is such a companion.

3.2. Which are the Structures Considered in Positive Logic?

If the Theory of Models for the full First Order Logic, with negation, consists in the study of elementary embeddings between structures, then the Theory of Models for Positive Logic should be the study of immersions between structures that are pc models of some h-inductive theories.

This class of models has an obvious intrinsic characterization: if M is a pc model of some h-inductive T , then it is a pc model of its own h-inductive theory, that is to say a pc model of its own h-universal theory. To save space and paper, we call ns (for *negatively sufficient*) this class of structures: in an ns M , according to Lemma 2, whenever a tuple \underline{a} of elements of M does not satisfy a certain positive formula $\varphi(\underline{x})$, it must satisfy some other positive

⁷ A proof of this fact was needed as a step for the Compactness Theorem.

⁸ By an inclusion of theories, we mean the reverse inclusion for their respective classes of models; that is, we do not distinguish between a theory and its axiomatizations.

formula $\psi(\underline{x})$ that M believes to be contradictory with it, that is to say that M satisfies the h-universal condition $\neg (\exists \underline{x}) \varphi(\underline{x}) \wedge \psi(\underline{x})$.

In the case of the full First Order Logic with negation, where one is content that any structure be a model of its own theory, the corresponding notion is tautologic; but it makes sense in Robinson's frame, where a structure is not necessarily an ec model of its own inductive theory.

Note that this property is sensitive to the language, since, as we have seen in the proof of Lemma 3, any structure M is obviously a ns model of its h-universal theory in the language $\mathcal{L}(M)$: if \underline{a} does not satisfy $\varphi(\underline{x})$, then $\varphi(\underline{x})$ and $\underline{x} = \underline{a}$ are incompatible.

Therefore, as was observed in Poizat [26] (introducing elementary extension in Positive Logic), any structure enters in the scope of Positive Logic at the price of an expansion of the language by individual constants, which is not such a mild operation as it is in the case of the logic with negation.

Lemma 4. (i) *For any structure, we can find a ns structure satisfying the same h-universal sentences.*

(ii) *If two ns structures satisfy the same h-universal sentences, then they are pc models of the same h-inductive theories.*

(iii) *An ns structure remains so after an expansion of the language by individual constants.*

Proof. (i) Any M can be continued into a pc model N of its own h-universal theory; since N is a continuation of M , it cannot satisfy more h-universal sentences than M .

(ii) Suppose that M is a pc model of the h-inductive theory T , i.e. of its h-universal companion T_u , and satisfies $\neg (\exists \underline{x}) \varphi(\underline{x}) \wedge \psi(\underline{x})$; $\varphi(\underline{x}) \wedge \psi(\underline{x})$ is in contradiction with $T_u \cup \text{Diag}^+(M)$, and there is a finite fragment $\delta(\underline{y})$ of $\text{Diag}^+(M)$ such that T_u implies $\neg (\exists \underline{x}, \underline{y}) \varphi(\underline{x}) \wedge \psi(\underline{x}) \wedge \delta(\underline{y})$, or, in other words, that $\varphi(\underline{x}) \wedge (\exists \underline{y}) \delta(\underline{y})$ contradicts $\varphi(\underline{x})$. Therefore, when M is ns, the fact that it is a pc model of T depends only of its h-universal theory: it means that M is a model of T_u , and that, for any pair of positive formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ which are contradictory in the sense of M , there is a third one, $\delta(\underline{y})$, such that $\neg (\exists \underline{y}) \delta(\underline{y})$ is untrue in M and that $\varphi(\underline{x})$ and $\psi(\underline{x}) \wedge \delta(\underline{y})$ are contradictory in the sense of T_u .

(iii) That \underline{b} does not satisfy $\varphi(\underline{a}, \underline{y})$ means that $\underline{a} \wedge \underline{b}$ does not satisfy $\varphi(\underline{x}, \underline{y})$. \square

Example 2. Lemma 4 (i) is no more true when we replace the h-universal theory by the h-inductive theory. Indeed, when the language is reduced to equality, any structure is a continuation of Terminus and can be continued into it, so that all the structures have the same h-universal theory; but Terminus is the only ns structure, and the only one to satisfy the h-inductive axiom $(\forall x, y) x = y$.

Corollary 5. *Consider a structure M immersed into a ns structure N ; then*

(i) *M is ns, and M and N are pc models of the same h-inductive theories;*

(ii) *M and N satisfy the same h-inductive sentences with parameters in M .*

- Proof.* (i) Suppose that \underline{a} in M does not satisfy the positive formula $\varphi(\underline{x})$, in the sense of M , but equivalently in the sense of N ; in N , but also in M , it satisfies some $\psi(\underline{x})$ such that $\neg(\exists \underline{x}) \varphi(\underline{x}) \wedge \psi(\underline{x})$ is true in N ; this h-universal sentence, being true in a continuation of M , is also true in M . Therefore M is ns. Moreover, since M is immersed in N , M and N have the same h-universal theory, and the conclusion comes from Lemma 4 (ii).
- (ii) M and N are also ns in the language $\mathcal{L}(M)$, and satisfy the same h-universal sentences in this language; the conclusion is by the same lemma. \square

3.3. Positively Model-Complete Theories

We say that an h-inductive T is *positively model-complete* if every model of T is pc, that is, if every homomorphism between models of T is pure; in this case, T is obviously equal to its Kaiser hull. Since the pc models form an h-inductive class, when T_k is not pos. model-complete they do not form an elementary class even in the sense of the Full Logic with negation.⁹

T is positively model-complete iff to every positive formula $\varphi(\underline{x})$ is associated another one, $\psi(\underline{x})$, such that T declares that each of them is the negation of the other. Indeed, if $\varphi(\underline{x})$ has no positive negation, then by compactness we can find a model M of T with a tuple \underline{a} not satisfying it, and also satisfying no positive formula contradictory to it; then $\varphi(\underline{a}) \cup \text{Diag}^+(M) \cup T$ is consistent, and this model M is not pc.

The reason for choosing the term is the same as the one given by Robinson: an h-inductive theory T is pos. model-complete precisely when, for each model M of T , the theory $T \cup \text{Diag}^+(M)$ is complete in the sense of full First Order Logic.

The following example of positive model-completeness is given in Kunzozhin [16]: if T_u is a finitely axiomatizable h-universal theory in a finite purely relational language, then the class of its dm models and the class of its pc models are elementary (the former being finitely axiomatizable).

The homomorphisms between models of a positively model-complete theory respect the satisfaction of the formulae of the full First Order Logic with negation: they are *elementary embeddings*. In a positively model-complete theory, every formula of the full First Order Logic is equivalent to a positive formula, so that positive model-completeness is stronger than the robinsonian notion (where every formula is equivalent only to an existential one).

The effect of positive Morleyisation is to transform any theory of the First Order Logic with negation into a positively model-complete h-inductive theory.

Example 3. The usual axioms of the theory of commutative rings, in the language $(+, -, \cdot, 0, 1)$, are positive inductive, with the exception of the essential

⁹ If you wish to ignore the fact that an h-inductive elementary class is axiomatized by h-inductive sentences, you can apply the argument of the next paragraph to the elementary theory of the pc models.

h -universal axiom $0 \neq 1$; atomic formulae are equivalent to polynomial equations with integer coefficients. Since any non-invertible element can be sent to 0 in some quotient of the ring, the dm rings are the fields; in them, the inequation $x \neq y$ is defined positively by the formula $(\exists z) z.(x - y) = 1$. The pc rings are the algebraically closed fields: this is the content of Hilbert's Nullstellensatz, which states that if K is an algebraically closed field, then any system of polynomial equations in n variables, with coefficients in K , has a solution in K provided that 1 does not belong to the ideal generated by these polynomials (if not the system cannot have a solution even in a continuation of K ; note that the obstruction to the existence of a solution is expressed by a positive condition on the coefficients of the system). So the theory T of algebraically closed fields is positively model-complete. Another way to see that is to eliminate the quantifiers by your favourite method, so that any formula is equivalent modulo T to a positive boolean combination of equations and inequations; then you replace the inequations by their positive (existential) expressions. The elimination of the quantifiers leads to the elimination of the negation.

3.4. Positive Logic and Robinson's Logic

Any h -inductive T is a fortiori inductive in the sense of Robinson, but there is no general reason why its pc models be ec , nor why its ec models be pc , as shows the example of the empty theory in the language of equality (the unique pc model is reduced to a point, the ec models are infinite). Also, as we have observed, the positive model-completeness of T is a stronger assumption than its model-completeness in the sense of Robinson; moreover, the amalgamation property for homomorphism (to be defined in the next section) is not the same thing as the amalgamation property for embeddings: none of them implies the other; similarly, the JCP defined in Sect. 3.6 is not the same thing as the JEP, its robinsonian version.

In other words, when we increase the language to render positive the negations of the atomic formulae, we may alter the properties of the theory.

But the following lemma shows that, if all the models of T are dm , then its pc models and its ec models are the same; T is $pos.$ model-complete if and only if it is model-complete, homomorphisms and embeddings being the same thing. In this case, naming the negations of atomic formulae is a benign operation, and we see that Robinson's inductive theories correspond exactly to the special case of h -inductive theories with the property described above.

Lemma 6. *Consider an h -inductive theory T and a positive formula $\varphi(\underline{x})$; if for every homomorphism h between models of T , $\varphi(\underline{a})$ is satisfied iff $\varphi(h(\underline{a}))$ is satisfied, then there is a positive formula which in T is the negation of φ .*

Proof. If $\varphi(\underline{x})$ has no $pos.$ negation, then, as was observed above, we can find a model M of T with some \underline{a} not satisfying $\varphi(\underline{x})$ and satisfying no positive formula contradictory to $\varphi(\underline{x})$ in T ; in this case, by compactness again, $T \cup \text{Diag}^+(A) \cup \varphi(\underline{a})$ is consistent, in contradiction with our hypothesis. \square

3.5. Spaces of Types, Amalgamation and Separation

Given an h -inductive T and a tuple of variables $\underline{x} = (x_1, \dots, x_n)$, a *complete n -type* is a maximal set of positive formulae $\varphi(\underline{x})$ which is consistent with T (or with any companion of T). Every type can be realized in some pc model; by Lemma 2, every tuple in a pc model realizes a complete type.

We put a topology on the sets $S_n(T)$ of types, by declaring that the type satisfying a given positive formula form a basic *closed* set; the general closed sets are therefore defined by arbitrary (infinite) conjunctions of formulae. We obtain in this way a compact set (English sense: if every finite subfamily of a family of closed sets has a non-empty intersection, then the total intersection of the family is non-empty), that does not necessarily satisfy Hausdorff's separation condition.

Théorème 20 in Ben Yaacov and Poizat [7] links the separation of the spaces of types to the following property: we say that the h -inductive theory T has the *h -Amalgamation Property* (for homomorphisms; in short, APh) if, whenever we consider two homomorphisms f from A to B and g from A to C , where A , B and C are models of T , then we can find a fourth model D of T , and homomorphisms f' from C to D and g' from B to D closing the diagram: $g' \circ f = f' \circ g$.

When the model A is pc, or more generally when the homomorphism f is pure, the amalgam is always possible since the positive diagram of B is finitely satisfiable in A .¹⁰ Therefore a pos. model-complete T has the APh.

If T has the h -Amalgamation Property, every h -inductive theory lying between T and T_k also has the APh, since D can be continued into a model of T_k . Another obvious remark: if T has the APh and M is a model of T , then $T \cup \text{Diag}^+(M)$, which is a theory in the language $\mathcal{L}(M)$, has the APh.

If T has the APh, a tuple \underline{a} extracted from any model M of T has a unique destiny: if f and g are two homomorphisms from M into some pc models of T , then $f(\underline{a})$ and $g(\underline{a})$ satisfy the same positive formulae. In fact, a syntactic characterisation of the APh is the following: for any pair $\varphi(\underline{x})$ and $\psi(\underline{x})$ of positive formulae which are contradictory in the sense of T , that is, in the sense of T_u , any \underline{a} from a model M of T satisfies in M a positive formula which is contradictory to one of them.

In particular M has a unique surjective dm continuation. And also:

Lemma 7. *In an h -inductive theory T with the h -Amalgamation Property, any model of T which is a substructure of a pc model of T is dm.*

Proof. Let A be a model of T which is a substructure of the pc model B of T ; if the atomic formula $\alpha(\underline{a})$ is not satisfied by some \underline{a} in A , then \underline{a} satisfies in B a positive formula $\varphi(\underline{x})$ which is contradictory to $\alpha(\underline{x})$; suppose that, in some

¹⁰ Belkamsi [8] introduces the notion of *amalgamation bases* for an arbitrary h -inductive theory T , which are its models over which can be amalgamated any pair of continuations within T . They form an inductive class, containing the pc models, and their h -inductive theory T_b is a companion of T ; in the case of rings, they are the rings with only one maximal ideal, axiomatized by $(\forall x)(\exists y) x.y = 1 \vee (1 - x).y = 1$.

continuation C of A , $\alpha(\underline{a})$ become true; then we cannot amalgamate B and C over A . \square

Consider two types p and q in $S_n(T) = S_n(\text{Tk})$ which are separated by two disjoint open sets; open sets being defined by possibly infinite disjunctions of formulae, this means that there are two positive formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ such that p does not satisfy the first, that q does not satisfy the second, but that every type satisfies one of them; in other words, the positive inductive axiom $(\forall \underline{x}) \varphi(\underline{x}) \vee \psi(\underline{x})$ is true in every pc model, and therefore belongs to Tk ; we shall say that T separates p and q if we can find formulae $\varphi(\underline{x})$ and $\psi(\underline{x})$ as above such that T implies that $(\forall \underline{x}) \varphi(\underline{x}) \vee \psi(\underline{x})$.

Let us introduce now a companion that does not appear in Robinson's context, and call *separant* of T the set T_s formed by its h -universal and positive inductive consequences.

Theorem 8. *An h -inductive theory T has the h -Amalgamation Property if and only if it separates each pair of distinct types; T has the APh if and only if T_s has the APh. The spaces of types of T satisfies Hausdorff's separation (in short: T is Hausdorff) if and only if Tk has the APh.*

Proof. Suppose that T separates the types, and consider A , B and C as above; we can assume that B and C are pc; each \underline{a} in A has no choice for its destiny, so that $f(A)$ and $g(A)$ satisfy the same positive formulae, and the theory $T \cup \text{Diag}^+(B) \cup \text{Diag}^+(C) \cup \{f(\underline{a}) = g(\underline{a}) / \underline{a} \in A\}$ is consistent, being finitely interpretable in C ; a model of it is an amalgam.

Reciprocally, suppose that T does not separate the distinct types p and q ; this means that we can find a model A of T with an \underline{a} satisfying only positive formulae which are common to p and q ; by the Lemme 16 of Ben Yaacov and Poizat [7], in some continuation B of A the image of \underline{a} is of type p , and in another continuation C its image has type q ; this is an obstacle to amalgamation.

We have observed that T and T_s separate the same pairs of types, and that separation in the spaces of types is the same thing than separation in the sense of Tk . \square

Example 4. In the language of infinitely many constants $c_0, c_1, \dots, c_n, \dots$ the h -universal theory formed by the axioms $c_m \neq c_n$ for $m < n$ is equal to its Kaiser hull, and does not have the APh, since any point distinct from all the c_n can be sent homomorphically to any of them; the space of 1-types consists in the c_n , the proper closed sets are finite, and no pair of types can be separated.

If we add to the language predicates $r(x), s_0(x), \dots, s_n(x), \dots$, and to the axioms the sentences saying that no c_n satisfies r , and that $s_n(x)$ is the negation of $x = c_n$, we obtain a theory T with APh whose models can be amalgamated into its unique pc model; the separant of Tk says in addition that r is not empty; it is weaker than Tk which says moreover that all the points in r are equal. The topologies on the spaces of types are easily seen to be Hausdorff: for instance, the space of 1-types is formed by the c_n , which are isolated, and

accumulate to the unique type satisfying $r(x)$. But T_k is not positively model-complete, since neither $r(x)$ nor $x = y$ have a positive negation (and observe that any positively model complete theory is equal to its separant).

In Robinson's setting, a well-known theorem of Lyndon states that if T is model-complete and if the set of its universal consequences has AP, then T eliminates quantifiers. The positivisation of this result is so strong that it loses any interest: any formula will be either tautological or antilogical.

Proposition 9. *If a consistent h -universal theory has the h -Amalgamation Property, then it has only one pc model; this model has only one point.*

Proof. Consider the free algebra M_2 for the functions of the language \mathcal{L} generated by the constants of \mathcal{L} plus two points x and y ; on M_2 , put the minimal structure, that is, no atomic formula is satisfied except the trivial equalities of terms $t(x,y,c) = t(x,y,c)$; M_2 can be sent homomorphically into any \mathcal{L} -structure, and moreover the choice of the images of x and y is arbitrary. Therefore, M_2 is a model of every consistent h -universal theory T .

If T has a pc model with two distinct points a and b , we can send (x,y) to (a,a) on one side, and to (a,b) on the other, making the amalgam impossible since (a,b) satisfies a positive formula incompatible with equality. So, if T has the AP h , all of its pc models have only one point, and since any pair of them can be amalgamated over M_2 , they must be isomorphic. If e is the unique element of the pc model of T , we have no choice for the interpretation of the individual constants and the functions (if T declares that $(\forall x) f(x) \neq x$, we cannot amalgamate); if r is a relation symbol in \mathcal{L} , $r(e,e, \dots e)$ is true unless T declares that $\neg(\exists \underline{x}) r(\underline{x})$. \square

We have introduced in Sect. 2.4 the pos. κ -saturated pc models of an h -inductive theory T , where κ is bigger than the cardinality of the language; when T has the AP h , we can give a combinatorial characterization of these big models. We say that M is κ -*extensible* if, whenever g is an homomorphism from A to B , where A and B are models of T of cardinality strictly less than κ , any homomorphism f of A into M can be extended to B ; in other words, there is an homomorphism f' from B to M such that f is the restriction to A of f' og.

When T has the AP h , we can iterate amalgams to construct κ -extensible continuations of any model of T , the cardinality limitations being the same as for the construction of pos. κ -saturated models.

Theorem 10. (i) *Every κ -extensible model of an h -inductive theory is pc and positively κ -saturated.*

(ii) *The converse holds when the theory has the AP h .*

(iii) *In fact, the AP h is necessary for the continuation of any model into a κ -extensible one.*

Proof. (i) Let M be a κ -extensible model of T , \underline{a} a tuple extracted from M , $\varphi(\underline{x})$ a positive formula, A a substructure of M containing \underline{a} , of cardinality less than κ , and B a pc continuation of A of cardinality less than κ . If the image \underline{a}' of \underline{a} in B satisfies φ , so does \underline{a} in M by extensibility; if \underline{a}' does not satisfy φ , it satisfy a ψ contradicting φ , and so does \underline{a} in M .

Let A be a substructure of M of cardinality less than κ , that we may assume to be a model of T ; if a set of positive formulae with parameters in A is finitely satisfiable in M , then it is satisfiable in some continuation of A which is a model of T of cardinality less than κ , and in M by extensibility.

- (ii) Assume that M is pc and pos. κ -saturated, and consider an homomorphism g between two models A and B of T of cardinal less than κ . Let f be an homomorphism of A into M , an N an amalgam of B and M over A ; denote by B' and A' the images of B and A in the amalgam; since M is pc, it is immersed in N , and the positive diagram of B' over A' is finitely satisfiable in M ; by saturation, it is satisfiable in M .
- (iii) The condition is obviously necessary for structures of cardinality less than κ , and is nothing but a matter of consistency of diagrams. \square

3.6. The Joint Continuation Property

We say that an h -inductive theory T has the *Joint Continuation Property* (in short, the JCP) if any two of its models can be simultaneous continued into a third one. We observe that if T has the JCP, then each of its companions has the JCP. Remark also that if T has a prime model (that is, a model that can be continued in any other model) and the AP ϕ , it has the JCP.

A positively model-complete T has the JCP if and only if it is complete in the sense of Full First Order Logic, since in this case the JCP means that any pair of models of T have a common elementary extension.

Proposition 11. (i) *An h -inductive theory has the JCP if and only iff any two of its pc models satisfy the same h -universal sentences if and only if they satisfy the same h -inductive sentences.*

- (ii) *If an h -inductive theory has the JCP and its Kaiser hull is not pos. model-complete, then none of its pc models is ω -saturated in the sense of the logic with negation.*

Proof. (i) If M and N are two models of our theory T , they can be continued into pc models M' and N' of T ; if M' and N' have the same h -universal theory, the union of their diagrams is consistent with T .

Reciprocally, suppose that T has the JCP; any two pc models of T can be simultaneously continued into a third, and satisfy the same h -inductive sentences by Corollary 5.

- (ii) Suppose that there is a positive formula $\varphi(\underline{x})$ without a positive negation in the sense of T_k , and consider a pc model M of T ; since M satisfies no more h -inductive sentences than the ones in T_k , we can find an elementary extension of M with a tuple \underline{a} satisfying $\varphi(\underline{a})$ but satisfying no positive formula contradictory to it; the type of \underline{a} is omitted in M . \square

Proposition 12. *Let T be an h -inductive theory.*

- (i) *The property “to have a common continuation which is a model of T ” defines an equivalence relation on the pc models of T .*
- (ii) *If T_M is the h -inductive theory of a pc model M of T , the pc models of T_M are the pc models of T which have a common continuation with M .*

Proof. (i) We must show transitivity; suppose that M and M' have a common continuation N' , and that M and M'' have a common continuation N'' , each of them model of T ; since M is pc, by compactness we can amalgamate N' and N'' above M into a model of T .

(ii) Let N be a pc model T continuing M and M' , the last one being a pc model of T ; since M is immersed in N , N satisfies T_M ; since M' is immersed in N , it is also a pc model of T_M .

If M' is a model of T_M , the theory formed by T and the positive diagrams of M and M' is consistent, so that M and M' have a common continuation N which is a model, and even a pc model, of T ; N is a model, and even a pc model, of T_M . If moreover M' is a pc model T , it is immersed in N , and a pc model of T_M . \square

Therefore, the JCP plays the role devoluted to completeness in Full First Order Logic. It insures the elementary equivalence of the positively ω -saturated pc models of T (see Sect. 2.4), and the uniqueness of the universal domain, that is, of the big κ -positively saturated pc model of cardinality at most (for a second time, yes!) κ , where κ is strongly inaccessible. Those of us who have no faith in big cardinals, but nevertheless crave for uniqueness, will take refuge not in the Buddha, not in the Dharma, not in the Sangha,¹¹ but in special models.

The JCP is quite a convenient hypothesis; when we do not assume it, we have to split the theory into its components, corresponding to the various h-universal theories of its pc models.

3.7. Bounded Theories

We say that a h-inductive theory T is *unbounded* if it has pc models of arbitrary large cardinality. A *bounded* (that is, not unbounded) theory may have infinite pc models: in the absence of a positive expression for the negation of equality, this does not contradict the Compactness Theorem.

Consider a positive formula $\varphi(x,y)$ which is in contradiction with the equality $x = y$; a *clique* for this formula is a subset A of a model of T such that any pair of distinct elements of A satisfy $\varphi(x,y) \vee \varphi(y,x)$. If the formula has finite cliques of an arbitrary large number of elements, then, by compactness, it has cliques of any infinite cardinality, and any pc model containing such a clique will be at least that big.

Reciprocally, by the Erdos-Rado Theorem, since in a pc model any pair of distinct elements must be different for some positive reason, an unbounded theory has necessarily a positive formula, incompatible with equality, which has unbounded cliques; and, in fact, a pc model of a bounded theory has no more than $2^{\text{card}(\mathcal{L})}$ elements.

Proposition 13. *A bounded h-inductive theory T with the JCP has, up to isomorphism, a unique pc model which is universal, continuing all the other models of T .*

¹¹ See the entry *kontchog sum* in the dictionary of Poizat [28].

Proof. The theory has a pc model M which embeds all the others; if M is not unique, it can be properly embedded into a copy M_1 of itself; M_1 is in turn properly embedded into M_2 , etc.; when we have constructed them we embed the M_n into a copy M_ω of M , and repeat ad libitum, ending with a pc model of a size higher than the cardinality of M : this is impossible. \square

We conclude by two simple examples of bounded theories; in the first, extracted from Poizat [27], the universal pc model is the only pos. ω -saturated pc model; in the second, on the contrary, all the pc models are ω -saturated.

Example 5. Let M be the segment $]0\ 1[$ of the rational numbers, equipped with their natural order; in the language \mathcal{L} of the strict order $<$, since $x \neq y$ can be expressed as $x < y \vee x > y$, we have positive quantifier elimination and the theory $\text{Tk}(M)$ is the familiar theory of a dense linear order without endpoints. But in the language \mathcal{L}' of the loose order \leq , nothing positive can force a cut to be filled by two distinct points, so that the h-inductive theory $\text{T}'k(M)$ is bounded, its universal pc model being the real segment $[0\ 1]$.

In this example, $\text{T}'k(M)$ is not the full first order theory of its pc models, since it is unable to express that the order is dense. In fact, by a simple back-and-forth argument, one sees that any two infinite loose linear orders satisfy the same h-inductive sentences; this remains true when we name elements, provided that the segments they bound be infinite.

Example 6. The language \mathcal{L} contains infinitely many relation symbols $e_n(x,y)$ and $e'_n(x,y)$; the axioms of T declare that e'_n is the negation of e_n , that each e_n is an equivalence relation, that e_{n+1} refines e_n , each class modulo the second being cut into two classes modulo the first. A model of T is pc, or dm, if and only if two elements congruent modulo all the e_n are equal.

3.8. Infinite Morleyisation, Dense Sets and Minimal Language

Ben Yaacov's philosophy is that all the properties of the pc ω -saturated models are recoverable from the spaces of types; by this, we mean that they can be reconstructed from the spaces of types "up to interpretation". Following Mustafin [20], we suggest to call *semantic properties* of an h-inductive theory, or more exactly of its Kaiser hull, those properties that depend only on the spaces of types, and not on the language. For instance the Amalgamation Property (for Tk) is semantic, since it means that the spaces of types are Hausdorff; the JCP is also semantic, since it means that two types are always compatible, in other words that each projection from $S_{m+n}(T)$ to $S_m(T) \times S_n(T)$ is surjective.

By compactness, a clopen subset of $S_n(T)$ must be defined by a formula, so that, when we deal with full First Order Logic, or equivalently with a positively model-complete theory, we can recover the theory and its models up to interpretation by assigning a name to every clopen subset of the spaces of types. But in Positive Logic formulae define closed sets that may not be open, so that we must expect that the canonical language will be associated to the closed sets. This is indeed the case thanks to the following process of *infinite morleyisation*: if T is an h-inductive theory and $F(\underline{x})$ is a closed set

of $S_n(T)$, defined by an infinite conjunction of positive formulae $\varphi_i(\underline{x})$, we add to the language a symbol for F , and to the axioms of T all the sentences $(\forall \underline{x}) F(\underline{x}) \Rightarrow \varphi_i(\underline{x})$ to form a theory T' ; then the following lemma, reproducing the Lemme 25 of Ben-Yaacov and Poizat [7], shows that T and T' have essentially the same ω -saturated pc models:

Lemma 14. *The pos. ω -saturated pc models of T' are the pos. ω -saturated pc models of T where $F(\underline{x})$ is interpreted as the conjunction of the $\varphi_i(\underline{x})$.*

Proof. In a pc model of T' , $F(\underline{x})$ must be interpreted as the conjunction of the $\varphi_i(\underline{x})$. Since any model of T can be transformed into a model of T' , the types of T are consistent with T' , and therefore an ω -saturated pc model of T' must be also ω -saturated for T ; therefore T and T' have the same spaces of types, so that any ω -saturated pc models can be transformed into an ω -saturated pc model of T' . \square

T' cannot be positively model-complete when F is not clopen, so that model-completeness is not a semantic property; but we can characterize semantically the morleyisations of model-complete theories as follows: each S_n has a Hausdorff totally disconnected topology, generated by clopen sets; equality is clopen; the projection from S_{n+1} onto S_n is open (that is, the image of an open set is open). The minimal language of the theory, up to interpretation, corresponds to the clopen sets of the spaces of types, and its models to the dense subsets of the positively ω -saturated pc models, which we define in the next paragraph.

If \underline{a} is a tuple of elements of some pc model M of T , it makes sense to speak of the types *over* \underline{a} , as the types of the theory $T(\underline{a})$, in the language $\mathcal{L}(\underline{a})$, obtained by adding to T all the positive formulae satisfied by \underline{a} . Since M is pc, every finite fragment of such a type is realized in M (if not, it would be in contradiction with a positive formula satisfied by \underline{a}). A basic open set being composed of the types that do not satisfy a certain positive formula, that is, of the types satisfying some other positive formula in contradiction with it, we see also that the n -types over M which are realized in M form a dense subset of the space $S_n(M)$. We shall call *dense set* any subset of a pc model of T having this property; since the notion of pc model is language-dependent, it cannot be characterized semantically by this topological property.

There is another case of existence of a minimal language: the projection on $S_m(T)$ of a clopen subset of $S_{m+n}(T)$ defined by a formula $\varphi(\underline{x}, \underline{y})$ is defined by $(\exists \underline{y}) \varphi(\underline{x}, \underline{y})$. Therefore, if we work in Robinson's setting, we have a minimal language corresponding to the projections of the clopen sets, and in this case also the pc models are the dense sets (since the test for existential closedness can be restricted to quantifier-free formulae).

By contrast, there is no minimal language in the following example.

Example 7. We consider the real segment $[0, 1]$ as a structure in the (unary) language \mathcal{L} containing a predicate for each of its closed subsets; let T be its h -inductive theory in this language; for each real a between 0 and 1, an axiom of T declares that $(\forall x) x \leq a \vee a \leq x$, and another that $(\exists x) x = a$, so that M

is the only pc model of T ; $S_1(T)$ is $[0, 1]$ with the usual topology, and $S_n(T)$ bears the product topology; note that the diagonal $x = y$, and also the order $x \leq y$, define closed subsets of $S_2(T)$.

To generate the topologies, we can take the sublanguage \mathcal{L}_2 of finite unions of closed segments with endpoints of the form $m/2^n$; we can also take the sublanguage \mathcal{L}_3 where the endpoints have the form $m/3^n$. The only subsets of M which are defined in both languages are M and \emptyset ; moreover, a subset of M is pc for the language \mathcal{L}_2 if and only if it contains all the rational numbers of the form $m/2^n$; in fact, when the language contains singletons, there are dense subsets which are not pc.

The conclusion is that, in the general situation, the notion of formula, and consequently the notion of pc model, are fragile: the positively ω -saturated pc models form the only secure raft on which we can stand. And in the case we find them too much language-dependent, we can jump into the Positive Universes of Poizat [26].

In our positive context, a *principal type*, which is the only one to satisfy a certain positive formula, must be distinguished from an *isolated type*, which is the only one not to satisfy a given positive formula $\varphi(\underline{x})$; an isolated type is principal, since it must satisfy a formula $\psi(\underline{x})$ contradictory to $\varphi(\underline{x})$, and is obviously the only type to satisfy $\psi(\underline{x})$, but the converse is not true, even for the minimal language in Robinson's context. The positive adaptation¹² of the Omitting Types Theorem is contained in Nurtazin [24]: *If T is an h -inductive theory in a countable language, and p_1, \dots, p_n, \dots is a sequence of non-principal types, there exists a pc model of T omitting all of them*; and various consequences are drawn from it.

We assume that our readers will be grateful to us for closing the section with two very simple examples.

Example 8. The language \mathcal{L} contains infinitely many constants $c_0, c_1, \dots, c_n, \dots$, a predicate symbol $r(x)$ and a binary relation symbol $i(x, y)$; the axioms for T' says that the c_n are pairwise distinct, that they do not satisfy r , that $i(x, y)$ is the negation of the equality and that r is infinite. We observe that this theory is complete in the sense of full First Order Logic, and that it satisfies the APh. Its pc models contain no point outside r except the c_i , contrarily to its model which are saturated for the logic with negation. Its spaces of positive 1-types is formed by the types of the c_i , which are isolated, and a type p satisfying r , which is not isolated: it is the accumulation point of the c_i ; this space is Hausdorff, as was expected from the APh.

When we drop r from the language, we obtain the familiar model-complete theory T of infinitely many constants: T and T' have the same space of types, and the pc models of T' are the ω -saturated models of T . This is an example of infinite morleyisation.

Example 9. This example is given in Robinson's setting (where languages contains implicitly the negations of the atomic formulae); \mathcal{L} has infinitely

¹² In robinsonian setting, but the extension to the general positive setting is obvious.

many individual constants $c_0, c_1, \dots, c_n, \dots, d_0, d_1, \dots, d_n, \dots, e_0, e_1, \dots, e_n, \dots$ and a binary relation symbol $r(x,y)$; when $r(x,y)$ is satisfied we say that x and y are partners.

The axioms of T say that $r(x,y)$ is symmetric and irreflexive, that any point has at most one partner, that the constants interpret distinct elements, that d_n and e_n are partners, and that the c_n have no partner.

T is a complete universal theory, and, in a saturated model of T , there are infinitely many points without partner and distinct from the c_i .

In an ec model of T , every element distinct from the c_i has a partner; T is not model-complete, as the fact that x has no partner cannot be expressed by an existential formula; T has the Amalgamation Property for embeddings, and one sees easily that its spaces of existential types are Hausdorff.

The inspection of the spaces of existential types shows that T is not an infinite Morleyisation of a model-complete theory, since $r(x,y)$ defines a clopen subset of $S_2(T)$, whose projection on $S_1(T)$ is not open.

3.9. Positive Jonsson Theories

Jerome Keisler (in Barwise [1], Ch. 2, Def. 6.1)¹³ was apparently the first to name *Jonsson theories* the theories in the full First Order language whose class of models satisfies the hypotheses, formulated (finding some inspiration in Fraïssé [11]) by Bjarni Jonsson [13,14], which allow the construction of an universal domain, that is a unique κ -homogeneous-universal structure of strongly inaccessible cardinality κ . The paradigm of such a Jonsson's class is given by the models of a complete theory in the full First Order Logic with negation, after Morleyisation.

We extend the notion to our positive context, and call *Positive Jonsson Theory* an h-inductive theory having the JCP and the h-Amalgamation Property.¹⁴ We recall that the h-Amalgamation Property for the Kaiser hull of the theory means that the spaces of types are Hausdorff; for instance, for any structure M , the theory $\text{Tk}(M)$ is Hausdorff if and only iff it is Jonsson.

A consequence of the Théorème 1 of Poizat [27] is that the universal model of a bounded Jonsson theory is its unique pos. \mathcal{L}^+ -saturated pc model; the two examples given in Sect. 3.7 are Jonsson.

In the case of a pos. Jonsson theory T , we can give a second combinatorial characterization of the big models, which is well-known in Robinson's setting

¹³ There is a disagreement between the English version of this definition and its Russian translation (Nauka, Moskva); in the latter, the theory itself has AP, but in the original the set of its universal consequences has AP. We follow the Russian version for three strong reasons; the first is that the sovietic logicians that we shall quote have worked under this definition; the second is our Proposition 9; the third is that we wish to absorb Robinson's context.

¹⁴ We drop the condition that the theory should have infinite models. In the absence of a negation for equality, this does not imply the existence of infinite pc models; the corresponding positive condition should be the existence of arbitrarily large pc models, but we see no good reasons to eliminate the case of a bounded universal model (a finite one in Robinson's context).

(see Mustafin [21]). Given a cardinal κ bigger than the cardinal of the language \mathcal{L} , we say that a model M of T is:

- κ -universal if any model of T of cardinal strictly less than κ can be continued into M ;
- κ -maximal if any substructure of M , which is a model of T of cardinal strictly less than κ , is *dm* (this condition is void in Robinson's context)
- κ -homogeneous if, given a pair of substructures $A \subseteq B$ of M which are models of T of cardinality strictly less than κ , A being a substructure of B , any homomorphism from A into M can be extended to an homomorphism from B into M .

Theorem 15. *If T is a positive Jonsson theory, a model of T is κ -extensible if and only if it is κ -universal, κ -maximal and κ -homogeneous.*

Proof. Suppose that M be κ -extensible. It is κ -homogeneous, since homogeneity is a special case of extensibility (restricted to substructures of M). Since it is *pc*, it is κ -maximal by Lemma 7. If A is a model of T of cardinality less than κ , by the JCP an homomorphic image of A exists in some continuation of M which is a model of T ; since M is *pc* and positively κ -saturated, such an image exists inside M .

For the converse, consider an homomorphism g between two models A and B of T of cardinal less than κ ; let f be an homomorphism from A into M , and denote by A' the image of A under f ; B and A' can be amalgamated into a model C of T of cardinal less than κ (note that we can take $C = B$ if f is an embedding); since A' is *dm*, it is a substructure of C ; by universality, we obtain an homomorphic image C_1 of C in M , and denote A_1 and B_1 the corresponding images of A and B ; since A' is *dm*, it is in fact isomorphic to A_1 , and by homogeneity the reverse isomorphism from A_1 to A' extends to an homomorphism from B_1 into M . \square

In the Qaraghandy School (which made a speciality of the determination of Jonsson theories of concrete structures: see Mustafin and Nurkhaidarov [22], Nurkhaidarov [23], Yeshkeyev [30]) these big models were called semantic models, especially the unique big model of strongly inaccessible cardinality κ . If the theory T_κ is not model-complete, the semantic models are never ω -saturated in the sense of First Order Logic with negation. (Jonsson theories with a model-complete Kaiser hull were called *perfect*.)

Their common Full First Order Theory was named the *center* of the Jonsson theory T by Tölendi Mustafin (see Mustafin [20]). Given an n -tuple \underline{a} of elements of a model M of T , all the images of \underline{a} in any continuation of M in a semantic model have the same (negative) type in the sense of the center, that was called the *central type* of \underline{a} . The central types form a dense subset in the space of n -types of the center.¹⁵

We have given many examples where T_κ is not model-complete but is nevertheless complete in the sense of full First Order Logic with negation. We

¹⁵ In fact, the term “central type” was introduced in Yeshkeyev [31] in a context of pairs of models.

have also seen that this is not the general case; an example of a completion of T_k is the theory of its *generic models*, which are a special kind of pc models obtained by a model-theoretic forcing à la Robinson. This forcing is defined in two versions; in the infinite version, the forcing conditions are the model of T , and since the positively ω -saturated models are generic, the forced theory will be the center; but in the finite version, where the forcing conditions are the finite fragments of the positive diagrams of the models of T ,¹⁶ we may obtain something different: for instance, in our Example 5, the central models have endpoints, but not the generic ones. For more details on forcing in Jonsson context, we refer to Yeshkeyev [31].

Mustafin [21] studies the companions of a Jonsson theory, and provides many examples and counterexamples. Since the JCP is preserved by companionship, a companion of an h-inductive Jonsson theory T is Jonsson provided that it has the APh; in particular, every h-inductive theory between T and T_k is Jonsson.

If T and T' are two companion h-inductive Jonsson theories, the theory $T \vee T'$, formed by sentences $\sigma \vee \sigma'$ where σ is in T and σ' is in T' , is also Jonsson, since its models are the models of T and the models of T' . In consequence, T has at most one minimal Jonsson companion; it may have none, since there is no general reason why the intersection of all of its Jonsson companions should have the APh.

In conclusion, we wish to draw the attention of our readers to one of the most mysterious Jonsson theories, the Theory of Groups.

Example 10. If we consider a group as a structure in the language of multiplication, inverse and unit, it can be continued into Terminus, the group reduced to the unit; since moreover Terminus can be embedded into any group, all the groups satisfy the same h-universal sentences, and Terminus is the only ns group.

To obtain something of interest, we add to the language the negation of the equality, and enter into Robinson's setting: the pc groups become then the familiar ec groups, which are the groups G such that any finite system of equations and inequations with parameters in G , which has a solution in some group embedding G , must have already a solution in G .

Thank to the Theorem of Highman, Neumann and Neumann, if G is ec two n -tuples \underline{a} and \underline{b} of elements of G which satisfies the same quantifier-free formulae, that is which satisfy the same equations and inequations, must be conjugated, and therefore have the same type. The existential types are described by equations and inequations; this does not mean quantifiers elimination ("every formula is equivalent to a quantifier-free one"), but only that the clopen sets in the space of types are associated to quantifier-free formulae. The ec groups are never negatively ω -saturated, since they do not contain unconjugated pairs of elements of infinite order.

In any case, the spaces of types are Hausdorff, and indeed groups can be amalgamated; T_k is unstable, because the formula $x.y = y.x$, whose negation

¹⁶ And countability of the language is assumed to insure the existence of generic models.

is by assumption (or convention) positive, has the Property of Independance. We remark in passing that $x.z = z.y$ defines a clopen set in S_3 whose projection in S_2 is not open.

The universal domains, or, equivalently, the spaces of types, are poorly known. For instance, an isolated type is a finitely generated group whose isomorphy type is determined by a finite number of equations and inequations satisfied by its generating system. It is not known if there are isolated types other than the finite groups (an infinite isolated group would provide a very simple example of a finitely axiomatisable strongly minimal theory; see Makowski [18]).

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