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# ELEMENTARY PARTICLE PHYSICS AND FIELD THEORY

## MINIMAL COHOMOLOGICAL MODEL OF A SCALAR FIELD ON A RIEMANNIAN MANIFOLD

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*Lagrangians of the field-theory model of a scalar field are considered as 4-forms on a Riemannian manifold. The model is constructed on the basis of the Hodge inner product, this latter being an analog of the scalar product of two functions. Including the basis fields in the action of the terms with tetrads makes it possible to reproduce the Klein–Gordon equation and the Maxwell equations, and also the Einstein–Hilbert action. We conjecture that the principle of construction of the Lagrangians as 4-forms can give a criterion restricting possible forms of the field-theory models.*

**Keywords:** Riemannian manifold, differential forms, Hodge operator, cohomological model, GRT, Klein–Gordon equation.

### 1. GEOMETRIC FUNDAMENTALS OF THE MODEL

To start with, let us consider some geometric objects and operators which will be used below in the construction of the model.

#### 1.1. Tetrad

Let $x^\mu$ be the coordinates in some global region of the Riemannian manifold $R^{1,3}$. In a small neighborhood of each point of $R^{1,3}$ it is possible to choose a local coordinate system $x^a$ in which the metric takes the simple form

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (1)$$

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Thus, at each point in $R^{1,3}$ the following objects are defined: $ds = \sqrt{\eta_{ab} dx^a dx^b}$, the elementary oriented areas $dx^a \wedge dx^b$, the volumes $dx^a \wedge dx^b \wedge dx^c$, and the 4-volume form $\Omega = \frac{1}{4!} \varepsilon_{abcd} dx^a \wedge dx^b \wedge dx^c \wedge dx^d$, where $\varepsilon_{abcd}$ is the completely asymmetric tensor ($\varepsilon_{0123} = 1$).

The relationship between the differentials of the local coordinate system $dx^a$ and the differentials $dx^\mu$ at the given point is defined by the coefficients $h^a_\mu$ and $h^\mu_a$ ($h^a_\mu h^\mu_b = \delta^a_b$, $h^a_\mu h^\nu_a = \delta^\nu_\mu$):

$$dx^a = h^a_\mu dx^\mu, \; dx^\mu = h^\mu_a dx^a . \tag{2}$$

We define the basis $dx^a$ independently at each point of the space $R^{1,3}$. Here, the coefficients $h^a_\mu$ and $h^\mu_a$ transform into functions of the global coordinate system $h^a_\mu(x^\mu)$ and $h^\nu_a(x^\mu)$. Since the differentials $dx^a$ at different points of space are independent, they can be considered as a tetrad of basis fields $h^a$ which are not necessarily holonomic [1]. We define the change of the basis form $h^a$ upon a displacement $\delta x^\mu$ as

$$\delta h^a(x^\nu) = -\Gamma^a_{b\mu}(x^\nu) h^b(x^\nu) \delta x^\mu , \tag{3}$$

where the coefficients $\Gamma^a_{b\mu}$ should be associated with the connection on $R^{1,3}$. Thus, the covariant derivative of the basis covector has the form

$$\nabla_\mu h^a = -\Gamma^a_{b\mu} h^b . \tag{4}$$

Now, if we have any covector prescribed in the basis $h^a$, for example,

$$\varphi = \varphi_a h^a , \tag{5}$$

then its variation under the infinitesimal displacement $\delta x^\mu$ will have the form

$$\delta\varphi = \partial_\mu \varphi_a h^a \delta x^\mu - \varphi_a \Gamma^a_{b\mu} h^b \delta x^\mu . \tag{6}$$

The quantity $\delta\varphi$ can be expanded over the basis at the point $x^\mu$:

$$\delta\varphi_a h^a = \partial_\mu \varphi_a h^a \delta x^\mu - \varphi_a \Gamma^a_{b\mu} h^b \delta x^\mu ,$$

and the covariant derivative can now be defined as

$$\nabla_\mu \varphi_a = \partial_\mu \varphi_a - \varphi_b \Gamma^b_{a\mu} .$$

The covector specified by Eq. (5) in the basis $dx^\mu$ associated with the global coordinate system will have the form

$$\varphi = \varphi_a h^a_\mu dx^\mu = \varphi_\mu dx^\mu .$$

Expression (6) can be rewritten in the form

$$\delta\varphi = \partial_\mu(\varphi_\nu h_a^\nu)h^a\delta x^\mu - \varphi_\nu h_a^\nu \Gamma_{b\mu}^a h^b \delta x^\mu = \partial_\mu\varphi_\nu dx^\nu \delta x^\mu + \varphi_\nu(\partial_\mu h_a^\nu - h_b^\nu \Gamma_{a\mu}^b)h_\lambda^a dx^\lambda \delta x^\mu .$$

We expand the left-hand side over $dx^\nu$:

$$\delta\varphi_\nu dx^\nu = \partial_\mu\varphi_\nu dx^\nu \delta x^\mu + \varphi_\nu(\partial_\mu h_a^\nu - h_b^\nu \Gamma_{a\mu}^b)h_\lambda^a dx^\lambda \delta x^\mu ,$$

introduce the notation

$$\Gamma_{\mu\nu}^\lambda = h_b^\lambda \Gamma_{a\mu}^b h_\nu^a - h_\nu^a \partial_\mu h_a^\lambda ,$$

and define the covariant derivative for covectors in the usual form:

$$\nabla_\mu \varphi_\nu = \partial_\mu \varphi_\nu - \varphi_\lambda \Gamma_{\nu\mu}^\lambda .$$

To be more correct, we should write the latter expression as

$$(\nabla_\mu \varphi)_\nu = \partial_\mu \varphi_\nu - \varphi_\lambda \Gamma_{\nu\mu}^\lambda ,$$

because the covariant derivative does not follow from $\varphi_\nu(x^\mu)$, but from the entire 1-form $\varphi$. Multiplying the latter expression by $dx^\nu$ (the basis 1-form), we obtain

$$(\nabla_\mu \varphi)_\nu dx^\nu = \partial_\mu \varphi_\nu dx^\nu - \varphi_\nu \Gamma_{\lambda\mu}^\nu dx^\lambda .$$

On the other hand,

$$\nabla_\mu(\varphi_\nu dx^\nu) = \nabla_\mu(\varphi_\nu)dx^\nu + \varphi_\nu \nabla_\mu(dx^\nu) ,$$

whence it follows that

$$\nabla_\mu dx^\nu = -\Gamma_{\lambda\mu}^\nu dx^\lambda . \qquad (7)$$

**1.2. Exterior calculus**

Expression (3) also makes it possible to introduce a Cartan structure on the cotangent bundle of $R^{1,3}$. We shall understand the $\delta$ procedure in Eq. (3) as the construction of an oriented area from the covector $h^a$:

$$dh^a(x^\mu) = -\Gamma_{b\mu}^a h^b \wedge h^c h_c^\mu ,$$

whence

$$dh^a(x^\mu) = -\frac{1}{2} f_{bc}^a h^b \wedge h^c ,$$

where $f^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}$ are the Cartan structure coefficients. It is now possible to define the action of the exterior differential on an arbitrary 1-form $\varphi = \varphi_a h^a$:

$$d\varphi = \partial_b \varphi_a h^b \wedge h^a - \frac{1}{2}\varphi_a f^a_{bc} h^b \wedge h^c .$$

In the basis $dx^\mu$ we will have

$$d\varphi = d(\varphi_\mu h^\mu_a h^a) = \partial_\nu (\varphi_\mu h^\mu_a) h^a_\lambda dx^\nu \wedge dx^\lambda - \frac{1}{2}\varphi_\mu h^\mu_a f^a_{bc} h^b_\nu h^c_\lambda dx^\nu \wedge dx^\lambda$$

$$= \partial_\nu \varphi_\lambda dx^\nu \wedge dx^\lambda + \varphi_\mu \left( \partial_\nu h^\mu_a h^a_\lambda - \frac{1}{2} h^\mu_a f^a_{bc} h^b_\nu h^c_\lambda \right) dx^\nu \wedge dx^\lambda .$$

After making the notational substitution

$$\frac{1}{2} T^\mu_{\nu\lambda} dx^\nu \wedge dx^\lambda = \left( \frac{1}{2} h^\mu_a f^a_{bc} h^b_\nu h^c_\lambda - \partial_\nu h^\mu_a h^a_\lambda \right) dx^\nu \wedge dx^\lambda \qquad (8)$$

in the second term, we obtain

$$d\varphi = \partial_\nu \varphi_\lambda dx^\nu \wedge dx^\lambda - \frac{1}{2}\varphi_\mu T^\mu_{\nu\lambda} dx^\nu \wedge dx^\lambda .$$

Here $T^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} - \Gamma^\mu_{\lambda\nu}$ is the usual Riemannian torsion $R^{1,3}$. On the other hand, the left-hand term of this latter expression can be considered directly in the basis $dx^\mu$:

$$d\varphi = d(\varphi_\mu dx^\mu) = \partial_\nu \varphi_\mu dx^\nu \wedge dx^\mu .$$

In contrast to the basis $h^a$, which can be nonholonomic, i.e., $dh^a \neq 0$, the differentials $dx^\mu$ are exact 1-forms and, consequently, $d^2 x^\mu = 0$. Thus, within the framework of the formalism considered here the torsion introduced above $T^\mu_{\nu\lambda} = 0$ and the relationship between the structure constants $f^a_{bc}$ and the tetrad coefficients follows from Eq. (8):

$$f^a_{bc} = h^\nu_b h^\mu_c (\partial_\mu h^a_\nu - \partial_\nu h^a_\mu) .$$

**1.3. Covariant derivative**

The existence of the metric prescribed by Eq. (1) implies the existence of a basis $h_a = \eta_{ab} h^b$ conjugate to $h^a$, such that

$$\left\langle h_a, h^b \right\rangle = \delta^b_a . \qquad (9)$$

Hence it follows that the basis on the tangent space associated with the coordinates $x^\mu$ should be expressed in terms of the same coefficients $h_\mu = h^a_\mu h_a$. Thus, expression (9) can be understood as an algebraic relation for the basis vector fields

$$\left[h_b, h_c\right] = f^a_{bc} h_a .$$

The covariant derivative on the tangent space can be introduced in an analogous fashion:

$$\nabla_\mu h_a = h_b \Gamma^b_{a\mu} . \tag{10}$$

For an arbitrary vector $\xi = \xi^a h_a$ we have

$$\delta\xi = \delta\xi^a h_a + \xi^a \delta h_a = \partial_\mu \xi^a \delta x^\mu h_a + \xi^c h_a \Gamma^a_{c\mu} \delta x^\mu ,$$

whence

$$(\nabla_\mu \xi)^a = \partial_\mu \xi^a + \xi^c \Gamma^a_{c\mu} .$$

### 1.4. A Hodge operator

Each $p$-form can be put in correspondence with the $(4-p)$-form dual to it [2] according to the rule

$$\frac{1}{p!}\omega_{a_1\ldots a_p} \to \frac{-1}{p!(4-p)!}\omega_{a_1\ldots a_p}\eta^{a_1 b_1}\cdots\eta^{a_p b_p}\varepsilon_{b_1\cdots b_p b_{p+1}\cdots b_4} .$$

We shall denote the corresponding operator – the Hodge operator – as $\hat{*}$:

$$\hat{*}\omega_p = \omega_{4-p}$$

or

$$\hat{*}\left(\frac{1}{p!}\omega_{a_1\ldots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p}\right) = \frac{-1}{p!(4-p)!}\omega_{a_1\ldots a_p}\eta^{a_1 b_1}\cdots\eta^{a_p b_p}\varepsilon_{b_1\cdots b_p b_{p+1}\cdots b_4} dx^{b_{p+1}} \wedge \ldots \wedge dx^{b_4} .$$

Double application of this operator gives

$$\hat{*}^2 \omega_p = -(-1)^{p(4-p)}\omega_p .$$

It is not hard to verify that the definition of the action of the Hodge operator in the basis $dx^\mu$ will be analogous:

$$\hat{*}\left(\frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\right) = \frac{-1}{p!(4-p)!}\omega_{\mu_1\ldots\mu_p}\eta^{\mu_1\nu_1}\cdots\eta^{\mu_p\nu_p}\varepsilon_{\nu_1\cdots\nu_p\nu_{p+1}\cdots\nu_4}\sqrt{-g}\, dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_4} .$$

**1.5. Divergence**

We define the divergence operator $\bar{d}$ acting in the space of *p*-forms in the standard way (in the theory of the exterior calculus the divergence operator is usually denoted as $\delta$, but in our case this symbol is already taken by the variation operation):

$$\bar{d} = \hat{*} d \,\hat{*} .$$

For the 1-forms $\phi = \phi_\mu dx^\mu$ we have

$$\bar{d}\varphi = \frac{1}{\sqrt{-g}} \partial_\nu \left( \varphi_\mu g^{\mu\nu} \sqrt{-g} \right).$$

For the 2-forms $\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu$ we have

$$\bar{d}\omega = \frac{1}{\sqrt{-g}} \partial_\sigma \left( \omega_{\mu\nu} g^{\mu\sigma} g^{\nu\tau} \sqrt{-g} \right) g_{\tau\chi}\, dx^\chi .$$

Note that by virtue of the nilpotence of the exterior differential $d^2 = 0$ we have nilpotence of the divergence operator $\bar{d}^2 = \hat{*} d \,\hat{*}\hat{*} d \hat{*} = \pm \hat{*}\, d^2 \hat{*} = 0$.

It is not hard to see that the combinations of operators $d\bar{d}$ and $\bar{d}d$ do not change the rank of the *p*-forms. For the space of $0$-forms $d\bar{d}f = d\,\hat{*}\, d\,\hat{*}\, f \equiv 0$ by virtue of the fact that $\hat{*} f$ is a 4-form.

$$\bar{d}df = \frac{1}{\sqrt{-g}} \partial_\nu \left( \partial_\mu f g^{\mu\nu} \sqrt{-g} \right) = g^{\mu\nu} \nabla_\mu \nabla_\nu f = \Box f \qquad (11)$$

is the Laplace operator of the function $f$. For the space of 1-forms $\phi = \phi_\mu dx^\mu$:

$$d\bar{d}\varphi = d\left[ \frac{1}{\sqrt{-g}} \partial_\nu \left( \varphi_\mu g^{\mu\nu} \sqrt{-g} \right) \right] = \partial_\chi \left[ \frac{1}{\sqrt{-g}} \partial_\nu \left( \varphi_\mu g^{\mu\nu} \sqrt{-g} \right) \right] dx^\chi ,$$

$$\bar{d}d\phi = \bar{d}\,\partial_\mu \phi_\nu dx^\mu \wedge dx^\nu = \bar{d}\,\frac{1}{2}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) dx^\mu \wedge dx^\nu$$

$$= \frac{1}{\sqrt{-g}} \partial_\sigma \left( (\partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu) g^{\mu\sigma} g^{\nu\tau} \sqrt{-g} \right) g_{\tau\chi} dx^\chi .$$

As is well known, on a flat manifold the combination $\bar{d}d + d\bar{d} = \nabla^2$ is the Laplace operator. It can be shown that in the considered case of a Riemannian manifold, for 1-forms,

$$(\bar{d}d + d\bar{d})\varphi = \left[ g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi_\chi \right] dx^\chi + \left[ (\nabla_\chi \nabla_\nu - \nabla_\nu \nabla_\chi) g^{\nu\mu} \varphi_\mu \right] dx^\chi ,$$

or

$$(\bar{d}d + d\bar{d})\varphi = \left[ g^{\mu\nu}\nabla_\mu \nabla_\nu \varphi_\chi \right] dx^\chi + R_{\nu\chi} g^{\nu\mu}\varphi_\mu dx^\chi , \tag{12}$$

where $R_{\nu\chi}$ is the Ricci tensor [2].

### 1.6. *H* product

We define the inner (scalar) product of *p*-forms by means of the Hodge operator as

$$\langle \omega * \phi \rangle = -\int \omega \wedge \hat{*}\, \phi \tag{13}$$

and we shall refer to it in what follows as the $H$ product. In particular, for the 0-forms $f$ and $g$

$$\langle f * g \rangle = \frac{1}{4!}\int fg\varepsilon_{\lambda\alpha\beta\gamma}\sqrt{-g}dx^\lambda \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma = \int fg\sqrt{-g}d\Omega ,$$

where $d\Omega = dx^0 dx^1 dx^2 dx^3$ is the elementary 4-volume.

For the 1-forms $\phi_1 = \phi_\mu dx^\mu$ and $\varphi_1 = \varphi_\mu dx^\mu$ we have

$$\langle \varphi_1 * \phi_1 \rangle = \frac{1}{3!}\int \varphi_\mu \phi_\nu g^{\nu\lambda}\varepsilon_{\lambda\alpha\beta\gamma}\sqrt{-g}\; dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma = \int \varphi_\mu \phi_\nu g^{\mu\nu}\sqrt{-g}d\Omega . \tag{14}$$

For the 2-forms $\omega_2 = (1/2)\omega_{\mu\nu}dx^\mu \wedge dx^\nu$ and $\sigma_2 = (1/2)\sigma_{\mu\nu}dx^\mu \wedge dx^\nu$ we obtain

$$\langle \omega_2 * \sigma_2 \rangle = \frac{1}{2}\int \omega_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}\sigma_{\alpha\beta}\sqrt{-g}d\Omega . \tag{15}$$

For the 3-forms $\Psi_3 = (1/3!)\Psi_{\alpha\beta\gamma}dx^\alpha \wedge dx^\beta \wedge dx^\gamma$ and $\Phi_3 = (1/3!)\Phi_{\lambda\mu\nu}dx^\lambda \wedge dx^\mu \wedge dx^\nu$ we obtain

$$\langle \Psi_3 * \Phi_3 \rangle = -\frac{1}{3!}\int \Psi_{\alpha\beta\gamma}\Phi_{\lambda\mu\nu} g^{\alpha\lambda} g^{\beta\mu} g^{\gamma\nu}\sqrt{-g}\Omega . \tag{16}$$

Note the symmetry of the $H$ product:

$$\langle \omega * \varphi \rangle = \langle \varphi * \omega \rangle .$$

From the differentiation rule for a product of forms (of ranks $p$ and $p+1$) we have

$$d(\omega_p \wedge \hat{*}\varphi_{p+1}) = d\omega_p \wedge \hat{*}\varphi_{p+1} + (-1)^p \omega_p \wedge d\,\hat{*}\,\varphi_{p+1}$$

$$= d\omega_p \wedge \hat{*}\varphi_{p+1} - (-1)^{(4-p)p}(-1)^p \omega_p \wedge \hat{*}\,\hat{*}\, d\,\hat{*}\,\varphi_{p+1} = d\omega_p \wedge \hat{*}\varphi_{p+1} - \omega_p \wedge \hat{*}\bar{d}\varphi_{p+1} .$$

Thus, taking Gauss's theorem into account, we can write

$$\langle d\omega_p * \varphi_{p+1} \rangle = \langle \omega_p * \bar{d}\varphi_{p+1} \rangle + \int_{\partial\Omega} \omega_p \wedge \hat{*}\varphi_{p+1} , \tag{17}$$

where $\partial\Omega$ is the boundary of the Riemannian manifold $R^{1,3}$. In particular, on a compact manifold $\langle d\omega_p * \varphi_{p+1}\rangle = \langle \omega_p * \bar{d}\varphi_{p+1}\rangle$.

## 2. SCALAR FIELD

### 2.1. A real scalar field

The $H$ square of the scalar field $\varphi = \varphi(x^\mu)$

$$\langle \varphi * \varphi \rangle = \int \varphi^2 \sqrt{|g|} d\Omega ,$$

and the $H$ square of its differential

$$\langle d\varphi * d\varphi \rangle = \int g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sqrt{|g|} d\Omega$$

allow us to represent the action of a real scalar field [3]

$$S_\varphi = \frac{1}{2} \int \left( g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m^2 \varphi^2 \right) \sqrt{|g|} d\Omega$$

in the form of a difference

$$S_\varphi = \frac{1}{2} \langle d\varphi * d\varphi \rangle - \frac{m^2}{2} \langle \varphi * \varphi \rangle . \tag{18}$$

Variation of this difference,

$$S_\varphi[\varphi + \delta\varphi] = \frac{1}{2} \int_\Omega d(\varphi + \delta\varphi) \wedge \hat{*} d(\varphi + \delta\varphi) - \frac{m^2}{2} \int_\Omega (\varphi + \delta\varphi) \wedge \hat{*} (\varphi + \delta\varphi)$$

$$\approx S[\varphi] + \int_\Omega d\delta\varphi \wedge \hat{*}\varphi - m^2 \int_\Omega \delta\varphi \wedge \hat{*}\varphi = S[\varphi] + 2\int_\Omega \delta\varphi \wedge \hat{*} \left\{ \bar{d}d - m^2 \right\} \varphi ,$$

leads to the usual Klein–Gordon equation in the form

$$(\bar{d}d - m^2)\varphi = 0 ,$$

or, with Eq. (11) taken into account,

$$(\Box - m^2)\varphi = 0 .$$

### 2.2. General form of the action of a scalar field

Within the framework of the proposed formalism, the general action of a scalar field can be represented in the form

$$S[\varphi]=\int_{\Omega} L_4(\varphi,\hat{*}\varphi,d\varphi,\hat{*}d\varphi,d\,\hat{*}\,d\varphi,\hat{*}d\,\hat{*}\,d\varphi,\ldots)\,,$$

where it is implied that the Lagrangian is some 4-form $L_4$. Taking into account the nilpotence of the exterior differential, $d^2=0$, there is only one way to construct new objects from the available 0-forms – by successive application of the exterior derivative and the Hodge operator (the rank of the obtained form is indicated in the lower row):

| $\hat{*}\varphi$ | $\varphi$ | $d\varphi$ | $\hat{*}d\varphi$ | $d\,\hat{*}\,d\varphi$ | $\hat{*}d\,\hat{*}\,d\varphi$ | $d\,\hat{*}\,d\,\hat{*}\,d\varphi$ | $\hat{*}d\,\hat{*}\,d\,\hat{*}\,d\varphi$ | $d\,\hat{*}\,d\,\hat{*}\,d\,\hat{*}\,d\varphi$ |
|---|---|---|---|---|---|---|---|---|
| 4 | 0 | 1 | 3 | 4 | 0 | 1 | 3 | 4 |

Hence it is clear that it is possible to distinguish the following types of objects of ranks 0, 1, 3, and 4:

| Rank | 0 | 1 | 3 | 4 |
|---|---|---|---|---|
| Objects | $(\hat{*}d)^{2n}\varphi$ | $\hat{*}(\hat{*}d)^{2n+1}\varphi$ | $(\hat{*}d)^{2n+1}\varphi$ | $\hat{*}(\hat{*}d)^{2n}\varphi$ |

In the notation employed here, Gauss's theorem has the form

$$\int_{\Omega} f\wedge\hat{*}dg=\int_{\Omega} dg\wedge\hat{*}f=\int_{\partial\Omega} g\wedge\hat{*}f-\int_{\Omega} g\wedge d\,\hat{*}\,f\,.$$

Thus, the quadratic action terms, containing combinations of rank 4, of the form $\hat{*}(\hat{*}d)^{2m}\varphi\wedge(\hat{*}d)^{2n}\varphi$ and $\hat{*}(\hat{*}d)^{2n+1}\wedge\varphi(\hat{*}d)^{2m+1}\varphi$ can always be transformed into the following form:

$$\int_{\Omega}\hat{*}(\hat{*}d)^{2m}\varphi\wedge(\hat{*}d)^{2n}\varphi=\int_{\Omega}\varphi\wedge(d\hat{*})^{2n}\,\hat{*}\,(\hat{*}d)^{2m}\varphi+\int_{\partial\Omega}\ldots=\int_{\Omega}\varphi\wedge\hat{*}(\hat{*}d)^{2(n+m)}\varphi+\int_{\partial\Omega}\ldots,$$

$$\int_{\Omega}\hat{*}(\hat{*}d)^{2n+1}\varphi\wedge(\hat{*}d)^{2m+1}\varphi=-\int_{\Omega}\varphi\wedge(d\hat{*})^{2m+1}\,\hat{*}\,(\hat{*}d)^{2n+1}\varphi+\int_{\partial\Omega}\ldots=-\int_{\Omega}\varphi\wedge\hat{*}(\hat{*}d)^{2(m+n)}\varphi+\int_{\partial\Omega}\ldots,$$

where all of the integrals over the surface of the 4-volume are denoted as $\int_{\partial\Omega}\ldots$ Consequently, if we discard integrals over the surface of the volume, then any action, quadratic in the scalar field, can be represented in the form of a superposition:

$$S=\sum_{k=0}C_k\int_{\Omega}\varphi\wedge\hat{*}(\hat{*}d)^{2k}\varphi\,. \quad (19)$$

### 2.3. Action for a tetrad

Let us consider the $H$ products of the basis 1-forms $h^a=h^a_\mu dx^\mu$. From Eq. (14) we have

$$\left\langle h^a * h^b\right\rangle=\int h^a_\mu h^b_\nu g^{\mu\nu}\sqrt{-g}\,d\Omega=\eta^{ab}\int\sqrt{-g}\,d\Omega\,.$$

An interesting combination is the sum of $H$ products of differentials and divergences of the tetrad fields $\langle \bar{d}h^a * \bar{d}h^b \rangle + \langle dh^a * dh^b \rangle$:

$$\left\langle \bar{d}h^a * \bar{d}h^b \right\rangle + \left\langle dh^a * dh^b \right\rangle = \left\langle h^a * (d\bar{d} + \bar{d}d)h^b \right\rangle .$$

According to Eq. (12), we have

$$(\bar{d}d + d\bar{d})h^a = [g^{\mu\nu}\nabla_\mu \nabla_\nu h^a_\chi]dx^\chi + R_{\nu\chi} g^{\nu\mu} h^a_\mu dx^\chi .$$

Within the framework of teleparallel gravitation $\nabla_\mu h^a_\nu = 0$, which ensures agreement of the Riemannian connection with the metric [1]. Thus, on the right-hand side of the latter expression only the term with the Ricci tensor survives. Convolution with the local metric $\eta_{ab}$ gives

$$\eta_{ab} \left\langle h^a * (d\bar{d} + \bar{d}d)h^b \right\rangle = \eta_{ab} \int h^a_\alpha R_{\nu\chi} g^{\nu\mu} h^b_\mu g^{\alpha\chi} \sqrt{-g}\, d\Omega = \int g_{\alpha\mu} R_{\nu\chi} g^{\nu\mu} g^{\alpha\chi} \sqrt{-g}\, d\Omega = \int R \sqrt{-g}\, d\Omega ,$$

where $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature tensor. Thus, the Einstein–Hilbert action [4] for the gravitational field

$$S_{\text{EH}} = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g}\, d\Omega$$

can be written in the form

$$S_{\text{EH}} = \frac{1}{16\pi G} \eta_{ab} \left( \left\langle \bar{d}h^a * \bar{d}h^b \right\rangle + \left\langle dh^a * dh^b \right\rangle - 2 \left\langle h^a * h^b \right\rangle \right).$$

**2.4. A complex scalar field**

Let the scalar field $\phi$ take values in $C^1$, so that $\phi = \phi^1 + i\phi^2$, or in $R^2$

$$\phi = (\phi^1, \phi^2) ,$$

with the flat metric $\tilde{\eta}_{AB} = \text{diag}(1,1)$. Let $\boldsymbol{e}_A$ be the basis of the *interior* space ( $A = 1,2$ ). We define the interior connection in analogy with Eq. (10):

$$\delta \boldsymbol{e}_A = \boldsymbol{e}_B \tilde{\Gamma}^B_{A\mu} \delta x^\mu . \qquad (20)$$

Assigning local $U(1)$ symmetry implies that

$$\tilde{\Gamma}^B_{A\mu} = S^B_A A_\mu(x^\mu) ,$$

where $S^B_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the generator of the $u(1)$ algebra.

Thus, the covariant derivative $D_\mu$, being a variant of the Fock–Ivanenko derivative [1], acts as follows:

$$D_\mu \phi_A = \partial_\mu \phi_A - S_A^B A_\mu (x^\mu) \phi_B .$$

In the complex representation $\phi = \phi_0 (x^\mu) e^{i\varphi(x^\mu)}$ (here $\varphi(x^\mu)$ is the phase) the covariant derivative has the form

$$D_\mu \phi = \partial_\mu \phi + i A_\mu \phi . \tag{21}$$

(In field theory, the covariant derivative usually includes the charge $e$ as a characteristic of the interaction with the field, $D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$. Here we set $e = 1$ since other force fields are not being considered.)

Generalization of the exterior calculus to complex-valued forms implies the substitution $\partial_\mu \to D_\mu$. A consequence of this is loss of the property of nilpotence of the exterior differential. Indeed,

$$d^2\phi = d(D_\mu \phi dx^\mu) = d(\partial_\mu \phi dx^\mu + i A_\mu \phi dx^\mu) = D_\nu (\partial_\mu \phi + i A_\mu \phi) dx^\nu \wedge dx^\mu$$

$$= \partial_\nu (\partial_\mu \phi + i A_\mu \phi) dx^\nu \wedge dx^\mu + i A_\nu (\partial_\mu \phi + i A_\mu \phi) dx^\nu \wedge dx^\mu = \frac{i}{2} F_{\mu\nu} \phi dx^\mu \wedge dx^\nu ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Conservation of invariance of the $H$ product with respect to the $U(1)$ group requires a refinement of this product, namely that instead of definition (13) we will have

$$\langle \omega * \phi \rangle = -\int \omega^\oplus \wedge \hat{*} \phi . \tag{22}$$

Here the symbol $\oplus$ must be understood as a generalization of the Hermitian conjugate $\dagger$, but acting only on the interior space. In particular, in the complex representation $\oplus$ reduces simply to the complex conjugate, and in the matrix representation, to transposition $\left( S_B^{A\oplus} = -S_B^A \right)$.

Thus, the action of a complex scalar field preserves the form of the action represented by expression (18) in the cohomological formulation. In explicit form, taking the introduced rule $\partial_\mu \to D_\mu$ into account along with definitions (21) and (22), the action acquires the standard form

$$S_\phi = (\partial_\mu \phi^* - i A_\mu \phi^*)(\partial^\mu \phi + i A^\mu \phi) - m^2 \phi^* \phi .$$

**2.5. Action of the electromagnetic field**

On the basis of Eq. (20) it is possible to write an expression for the differential of the basis vector

$$d\boldsymbol{e}_A = \boldsymbol{e}_B S_A^B A_\mu (x^\mu) dx^\mu . \tag{23}$$

For the double differential we have

$$d^2 \boldsymbol{e}_A = d\boldsymbol{e}_B S_A^B A_\mu (x^\mu) \wedge dx^\mu + \boldsymbol{e}_B S_A^B \partial_\nu A_\mu (x^\mu) dx^\nu \wedge dx^\mu$$

$$= \boldsymbol{e}_C S^C_B A_\nu S^B_A A_\mu dx^\nu \wedge dx^\mu + \boldsymbol{e}_B S^B_A \partial_\nu A_\mu dx^\nu \wedge dx^\mu .$$

It is obvious that the first term is identically equal to zero; hence, we finally obtain

$$d^2 \boldsymbol{e}_A = \frac{1}{2} \boldsymbol{e}_B S^B_A F_{\mu\nu} dx^\mu \wedge dx^\nu . \tag{24}$$

Bearing expressions (15) and (24) in mind, it is possible to represent the action of the electromagnetic field as

$$\tilde{\eta}^{AB} \langle d^2 \boldsymbol{e}_A * d^2 \boldsymbol{e}_B \rangle = -\frac{1}{2} \tilde{\eta}^{AB} \int \boldsymbol{e}_C S^C_A F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} \boldsymbol{e}_D S^D_B F_{\alpha\beta} \sqrt{-g}\, d\Omega = \frac{1}{2} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d\Omega .$$

Note that the *H* square of the first differentials of the basis vectors $\tilde{\eta}^{AB} \langle d\boldsymbol{e}_A * d\boldsymbol{e}_B \rangle$ is proportional to $A_\mu A^\mu$ by virtue of Eq. (23). Such a term is not included in the action of charged particles and the electromagnetic field by virtue of the masslessness of photons. Thus, the action of the electromagnetic field can finally be written in the form

$$S_{EM} = -\frac{1}{8\pi} \tilde{\eta}^{AB} \langle d^2 \boldsymbol{e}_A * d^2 \boldsymbol{e}_B \rangle .$$

## CONCLUSIONS

At the present time, a huge number of papers have been published, dedicated to cohomological models on simplectic manifolds, which primarily investigate problems of quantization of dynamical systems and problems of stability and chaos. Therefore, the use of cohomological theory to construct field-theory models on Riemannian manifolds also holds much promise. In the present work it has been shown that the action of the simplest model *scalar field* + *electromagnetic interaction* + *gravitation* can be represented in the form of a superposition of *H* squares of the scalar field, the basis fields, and their differentials:

$$S = \frac{1}{2} \langle d\phi * d\phi \rangle - \frac{m^2}{2} \langle \phi * \phi \rangle - \frac{1}{8\pi} \tilde{\eta}^{AB} \left\langle d^2 \boldsymbol{e}_A * d^2 \boldsymbol{e}_B \right\rangle + \frac{1}{16\pi G} \eta_{ab} \left( \left\langle dh^a * dh^b \right\rangle + \left\langle \bar{d}h^a * \bar{d}h^b \right\rangle \right) .$$

It is not hard to show that observance of the principle of construction of the Lagrangian as a 4-form in Riemannian space imposes restrictions on the possible form of the action of a field-theory model. This can serve as an additional criterion of the geometrical *reasonableness* of the model, on an equal level with its invariance with respect to the Lorentz and gauge transformations.

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