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Hardy-Littlewood theorem for series with general monotone coefficients

In this work we study trigonometric series with general monotone coefficients. Also, we consider $L_q\varphi(L_q)$ space. In particular, when $\varphi(t) \equiv 1$ the space $L_q\varphi(L_q)$ coincides with L_q. Well known the theorem of Hardy and Littlewood about trigonometric series with monotone coefficients. Also known various generalizations of this theorem. In 1982 this theorem was generalized by M.F. Timan for the spaces $L_q\varphi(L_q)$. And in 2007 S.Tikhonov proved Hardy-Littlewood theorem for trigonometric series with general monotone coefficients. In this work we have generalized Hardy-Littlewood theorem for Fourier series of functions $f \in L_q\varphi(L_q)$ with general monotone coefficients. Also, obtained upper-bound estimate of best approximation of functions $f \in L_q$ through its Fourier's coefficients which are general monotone.

Keywords: trigonometric series, Hardy-Littlewood theorem, general monotone sequences, convergence, Fourier's coefficients.

Let $L_q(0, 2\pi)$, $1 \le q < +\infty$ denotes the space of all 2π- periodic, measurable by Lebesgue functions $f(x)$, for which

$$\|f\|_q = \left(\int_0^{2\pi} |f(x)|^q \, dx\right)^{\frac{1}{q}} < +\infty.$$

Through $E_n(f)_q$ we will designate the best approximation of a function $f \in L_q$ by trigonometrical polynomials of total degree n in the metric of spaces L_q:

$$E_n(f)_q = \inf_{T_n} \|f - T_n\|_q.$$

Let the function $\varphi(t)$ satisfies the following conditions [1]:
a) $\varphi(t)$ is an even, non-negative, non-decreasing on $[0, +\infty)$;
b) $\varphi(t^2) \le C\varphi(t)$, $t \in [0, \infty)$, $C \ge 1$;
c) $\frac{\varphi(t)}{t^\varepsilon} \downarrow$ on $(0, +\infty)$ for some $\varepsilon > 0$.
Measurable, 2π-periodic function $f \in L_q\varphi(L_q)$, if

$$\int_0^{\pi} |f(x)|^q \, \varphi\left(|f(x)|^q\right) dx < +\infty.$$

In particular, when $\varphi(t) \equiv 1$ the space $L_q\varphi(L_q)$ coincides with L_q.

We consider the series

$$\sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

and denote by $f(x)$ the sum of this series.

Definition [2]. The sequence of numbers $\{a_n\}$ is said to be general monotone, or $\{a_n\} \in GMS$, if the relation

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \le C|a_n|$$

holds for all $n \ge 1$, where the constant C is independent of n.

The set of all numerical sequences $\{a_n\}$ such that $a_n \downarrow 0$, $n \to \infty$, is denoted by MS. It is known that $MS \subset GMS$.

We give the following well-known theorem of Hardy-Littlewood

Theorem A [2]. Let $\{a_n\} \in MS$. A necessary and sufficient condition that the function $f(x) \in L_q$, $1 < q < +\infty$, is that

$$\sum_{n=1}^{+\infty} n^{q-2} \cdot a_n^q < +\infty.$$

In 1982 this theorem was generalized by M.F.Timan [1] for the spaces $L_q\varphi(L_q)$.

Theorem B. Let $\varphi(t)$ satisfies conditions a)-c) and $f(x) \in L_1$ is an even function with Fourier series $\sum\limits_{n=1}^{+\infty} a_n \cos nx$, where $\{a_n\} \in MS$. A necessary and sufficient condition that the function $f \in L_p\varphi(L_p)$ for some $p > 1$, is that

$$\sum_{n=1}^{+\infty} n^{p-2} \cdot a_n^p \varphi(n) < +\infty.$$

In 2007 S.Tikhonov [2] proved the following theorem.

Theorem C. Let $\{a_n\}$ be a positive sequence and $\{a_n\} \in GMS$. A necessary and sufficient condition that the function f should belong to L_q, $1 < q < +\infty$, is that inequality

$$\sum_{n=1}^{+\infty} n^{q-2} \cdot a_n^q < +\infty$$

holds.

Our main goal is to prove the theorem of Hardy and Littlewood for the Fourier series of a function $f \in L_q\varphi(L_q)$, the coefficients are generally monotonous.

To obtain the main result we need the following Lemma.

Lemma. Let $f(x) = \sum\limits_{n=1}^{+\infty} a_n \cos nx$, where positive sequence $\{a_n\} \in GMS$ and for some q, $1 < q < +\infty$ converges the series $\sum\limits_{n=1}^{+\infty} n^{q-2} \cdot a_n^q$.

Then, the following inequality holds

$$E_n(f)_q \leq C \left[a_n (n+1)^{1-\frac{1}{q}} + \left(\sum_{k=n+1}^{+\infty} k^{q-2} \cdot a_k^q \right)^{\frac{1}{q}} \right], n = 1, 2, ...$$

Proof. From the properties of the best approximation and norms, we have

$$E_n(f)_q \leq \|f - S_n(f)\|_q = \|f - S_n(f) + a_n \cdot D_n(\cdot) - a_n \cdot D_n(\cdot)\|_q \leq$$
$$\leq \|f - S_n(f) + a_n \cdot D_n(\cdot)\|_q + a_n \|D_n(\cdot)\|_q, \quad (2)$$

where $S_n(f)$ is the partial sum of series (1), $D_n(x) = \sum\limits_{k=1}^{n} \cos kx$ is the Dirichlet kernel.

For the Dirichlet kernel is known the following inequality [2]

$$\|D_n(\cdot)\|_q \leq C \cdot n^{1-\frac{1}{q}}. \quad (3)$$

To estimate the first term in (2) we use the theorem C. Because the sequence of coefficients of the series

$$f(x) - S_n(f)(x) + a_n \cdot D_n(x) = \sum_{k=1}^{+\infty} b_k \cdot \cos kx$$

belongs to GMS. Indeed, for

$$b_k = \begin{cases} a_n, & k = 1, ..., n; \\ a_k, & k = n+1, ... \end{cases}$$

we have

$$\sum_{k=m}^{2m-1} |b_k - b_{k+1}| = 0 < C \cdot b_m, \text{ at } 2m-1 \leq n;$$

$$\sum_{k=m}^{2m-1} |b_k - b_{k+1}| = \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \le C \cdot a_m = C \cdot b_m, \text{ at } m \ge n.$$

If $m < n < 2m-1$, then we assume that $b_k = a_n$, $k = m, ..., m+s$ and $b_k = a_k$, $k = m+s+1, ..., 2m-1$, where s is natural number. Then

$$\sum_{k=m}^{2m-1} |b_k - b_{k+1}| = \sum_{k=m}^{m+s-1} |b_k - b_{k+1}| + \sum_{k=m+s}^{2m-1} |b_k - b_{k+1}| =$$

$$= \sum_{k=m+s}^{2m-1} |a_k - a_{k+1}| < \sum_{k=m+s}^{2(m+s)-1} |a_k - a_{k+1}| \le C \cdot a_{m+s} \le C \cdot b_m.$$

Therefore, by theorem C

$$\|f - S_n(f) + a_n \cdot D_n(\cdot)\|_q \le C \left(a_n^q \sum_{k=1}^{n} k^{q-2} + \sum_{k=n+1}^{\infty} a_k^q \cdot k^{q-2} \right)^{\frac{1}{q}} \le$$

$$\le C \cdot a_n (n+1)^{1-\frac{1}{q}} + \left(\sum_{k=n+1}^{\infty} a_k^q \cdot k^{q-2} \right)^{\frac{1}{q}}. \tag{4}$$

Now, using inequalities (3) and (4), from (2) we have

$$E_n(f)_q \le C \left[a_n (n+1)^{1-\frac{1}{q}} + \left(\sum_{k=n+1}^{+\infty} k^{q-2} \cdot a_k^q \right)^{\frac{1}{q}} \right], n = 1, 2, ...$$

This completes the proof of Lemma.

Now we prove the main result:

Theorem. Let the function $\varphi(t)$ satisfies the conditions a)-c), and $f(x) \in L_1$ is an even function with Fourier series $\sum\limits_{n=1}^{+\infty} a_n \cos nx$, where $\{a_n\}$ is positive sequence, and $\{a_n\} \in GMS$.

A necessary and sufficient condition that the function f should belong to $L_q\varphi(L_q)$, $q > 1$, is that

$$\sum_{n=2}^{+\infty} n^{q-2} \cdot a_n^q \cdot \varphi(n) < +\infty. \tag{5}$$

Proof. Suppose inequality (5) holds. Then, from the properties of the function $\varphi(t)$ converges also following series

$$\sum_{n=2}^{+\infty} n^{q-2} \cdot a_n^q.$$

So, by theorem 4.2 of [2] $f \in L_q$. Then applying the Lemma, the inequality of Hardy [1] and the properties of the function $\varphi(t)$, we have for $1 < p_0 < q < +\infty$:

$$\sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \cdot E_n^q(f)_{p_0} \le C \cdot \sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \cdot \left[a_n (n+1)^{1-\frac{1}{p_0}} + \left(\sum_{k=n+1}^{+\infty} k^{p_0-2} \cdot a_k^{p_0} \right)^{\frac{1}{p_0}} \right]^q \le$$

$$\le C \cdot \sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \cdot a_n^q n^{q\left(1-\frac{1}{p_0}\right)} + C \cdot \sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2} \cdot \varphi(n) \left(\sum_{k=n+1}^{+\infty} k^{p_0-2} \cdot a_k^{p_0} \right)^{\frac{q}{p_0}} \le$$

$$\le C \cdot \sum_{n=1}^{+\infty} n^{q-2} \cdot \varphi(n) \cdot a_n^q + C \cdot \sum_{n=1}^{+\infty} n^{-\left(2-\frac{q}{p_0}\right)} \left(\sum_{k=n+1}^{+\infty} k^{p_0-2} \cdot a_k^{p_0} \varphi^{\frac{p_0}{q}}(k) \right)^{\frac{q}{p_0}} \le$$

$$\le C\cdot\sum_{n=1}^{+\infty} n^{q-2}\cdot\varphi(n)\cdot a_n^q + C\cdot\sum_{n=1}^{+\infty} n^{\frac{q}{p_0}-2}\left(n\cdot a_n^{p_0}\varphi^{\frac{p_0}{q}}(n)\cdot n^{p_0-2}\right)^{\frac{q}{p_0}}\le$$

$$\le C\cdot\sum_{n=1}^{+\infty} n^{q-2}\cdot a_n^q\varphi(n) < +\infty.$$

Hence by theorem 3.3 of [1] we have

$$\int_0^\pi |f(x)|^q\,\varphi\left(|f(x)|^q\right)dx < +\infty.$$

Now let us prove the opposite. Let $f(x)\in L_q\varphi(L_q)$. Then by theorem 13.1 of [3; 54] we have

$$\int_0^\pi |f(x)|^q\,\varphi\left(|f(x)|^q\right)dx = \int_0^\pi \left(f^*(t)\right)^q\varphi\left(\left(f^*(t)\right)^q\right)dt < +\infty,$$

where f^* — is non-increasing rearrangement of f.

Let

$$f_1(x)=\int_0^x f(u)du \quad\text{and}\quad f_2(x)=\int_0^x f_1(u)du,\ \text{ for }\ x\in(0,\pi).$$

Also for $x\in(0,\pi)$ we denote

$$f_3(x)=\int_0^x f^*(u)du \quad\text{and}\quad f_4(x)=\int_0^x f_3(u)du.$$

In [4] proved that for $\frac{\pi}{4(n+1)}\le x\le\frac{\pi}{4n}$ the following inequality is satisfied

$$f_4(x)\ge|f_2(x)|\ge f_2(x)\ge\frac{C}{n^2}\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k.$$

Next, arguing as in [4], we have

$$\sum_{n=1}^{+\infty} n^{q-2}\cdot a_n^q\varphi(n)\le C\cdot\sum_{n=1}^{+\infty} n^{q-2}\cdot\varphi(n)\left(\frac{2}{n}\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k\right)^q=$$

$$=C\cdot\sum_{n=1}^{+\infty} n^{-2}\cdot\varphi(n)\left(\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k\right)^q\le C\cdot\sum_{n=1}^{+\infty} n^{2q-2}\cdot\varphi(n)\min_{\frac{\pi}{4(n+1)}\le x\le\frac{\pi}{4n}}\left(f_4(x)\right)^q\le$$

$$\le C\cdot\sum_{n=1}^{+\infty}\varphi(n)\frac{4n(n+1)}{\pi}\int_{\frac{\pi}{4(n+1)}}^{\frac{\pi}{4n}}\left(\frac{\pi}{4x}\right)^{2q-2}\left(f_4(x)\right)^q dx\le$$

$$\le C\cdot\sum_{n=1}^{+\infty}\int_{\frac{\pi}{4(n+1)}}^{\frac{\pi}{4n}}\left(\frac{\pi}{4x}\right)^{2q}\varphi\left(\frac{\pi}{x}\right)\left(f_4(x)\right)^q dx\le$$

$$\le C\cdot\int_0^\pi x^{-2q}\varphi\left(\frac{\pi}{x}\right)\left(\int_0^x f_3(u)du\right)^q dx=$$

$$=C\int_0^\pi x^{-2q}\varphi\left(\frac{\pi}{x}\right)\left(\int_0^x\left(\int_0^u f^*(t)dt\right)du\right)^q dx\le$$

$$\le C\int_0^\pi x^{-q}\left(\int_0^x\left(\int_0^u \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)\frac{du}{u}\right)^q dx\le$$

$$\le C\int_0^\pi x^{-q+1}\left(\int_0^x\left(\int_0^u \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)\frac{du}{u}\right)^q \frac{dx}{x}=$$

$$= C\int_0^{\pi}\left(x^{-1+\frac{1}{q}}\int_0^x\left(\int_0^u \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)\frac{du}{u}\right)^q\frac{dx}{x}\leq$$

$$\leq C\int_0^{\pi}\left(x^{-1+\frac{1}{q}}\int_0^x \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)^q\frac{dx}{x}=$$

$$= C\int_0^{\pi}\left(\frac{1}{x}\int_0^x \varphi^{\frac{1}{q}}\left(\frac{\pi}{t}\right)f^*(t)dt\right)^q dx\leq C\cdot\int_0^{\pi}\varphi\left(\frac{\pi}{t}\right)\cdot\left(f^*(t)\right)^q dt\leq$$

$$\leq C\cdot\int_0^{\pi}\left(f^*(t)\right)^q\varphi\left(\left(f^*(t)\right)^q\right)dt<+\infty.$$

This completes the proof of Theorem.

Remark. The proved Theorem is extension of the Theorem B. Also, at $\varphi(t)\equiv 1$ the Theorem C follows from the proved Theorem.

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С.Бітімхан

Жалпы монотонды коэффициентті қатарлар үшін Харди-Литтлвуд теоремасы

Мақалада жалпы монотонды коэффициентті тригонометриялық қатарлар зерттелді. Сонымен бірге $L_q\varphi(L_q)$ кеңістігі қарастырылды. Дербес жағдайда $\varphi(t)\equiv 1$ болғанда $L_q\varphi(L_q)$ кеңістігі L_q кеңістігімен беттеседі. Монотонды коэффициентті тригонометриялық қатарлар үшін Харди мен Литтлвуд теоремасы жақсы белгілі. Сондай-ақ теореманың әртүрлі жалпыламалары да белгілі. 1982 ж. осы теореманы М.Ф.Тиман $L_q\varphi(L_q)$ кеңістігі үшін жалпылады, ал 2007 ж. С.Тихонов Харди-Литтлвуд теоремасын жалпы монотонды коэффициентті тригонометриялық қатарлар үшін дәлелдеді. Бұл жұмыста Харди-Литтлвуд теоремасын $f\in L_q\varphi(L_q)$ функциясының коэффициенттері жалпы монотонды болатын Фурье қатарлары үшін жалпыланды. Сонымен бірге $f\in L_q$ функциясының ең жақсы жуықтауының жоғарыдан бағалауын оның жалпы монотонды болатын Фурье коэффициенттері арқылы алынды.

Кілт сөздер: тригонометриялық қатарлар, Харди-Литтлвуд теоремасы, жалпы монотонды тізбектер, жинақтылық, Фурье коэффициенттері.

С.Битимхан

Теорема Харди-Литтлвуда для рядов с обобщенно-монотонными коэффициентами

В статье исследованы тригонометрические ряды с обобщенно-монотонными коэффициентами. Также рассмотрено пространство $L_q\varphi(L_q)$. В частности, когда $\varphi(t) \equiv 1$, пространство $L_q\varphi(L_q)$ совпадает с L_q. Хорошо известна теорема Харди и Литтлвуда о тригонометрических рядах с монотонными коэффициентами. Также известны различные обобщения этой теоремы. В 1982 г. М.Ф.Тиман обобщил эту теорему для пространства $L_q\varphi(L_q)$, а в 2007 г. С.Тихонов доказал теорему Харди-Литтлвуда для тригонометрических рядов с обобщенно-монотонными коэффициентами. В данной работе обобщили теорему Харди-Литтлвуда для рядов Фурье функции $f \in L_q\varphi(L_q)$ с обобщенно-монотонными коэффициентами. Также получена верхняя оценка наилучшего приближения функции $f \in L_q$ через её коэффициенты Фурье, которые являются обобщенно-монотонными.

Ключевые слова: тригонометрические ряды, теорема Харди-Литтлвуда, обобщенно-монотонные последовательности, сходимость, коэффициенты Фурье.

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