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The structure of lattices of positive existential formulae of $(\Delta - PJ)$-theories

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ABSTRACT: This article is related to one of the main branches of mathematical logic, model theory, and more precisely to what is called eastern model theory. This part of model theory is concerned with the study of incomplete inductive theories and more precisely Jonsson theories and some of their positive generalizations. It examines the model-theoretical properties of positive Jonsson theories. In particular, the lattice of special formulae is considered. In the study of complete theories one of the main methods is to use the properties of a topological space $S_n(T)$. In the case of positive Jonsson theory, we can consider the lattice $E_n^+(T)$ of existential formulae, which is a sublattice of the Boolean algebra $F_n^+(T)$. The main aim of this article is to develop the basic concepts and methods of that part of model theory which will provide an opportunity for fruitful studies of Jonsson theories and some of its positive generalizations. Our technique is standard in the study of incomplete theories. The method consists of the translation of the elementary properties of the centre of a Jonsson theory into the theory itself.

KEYWORDS: Jonsson theories, perfect Jonsson theories, lattice of existential formulae, isomorphic embedding, Δ-continuation, Δ-immersion, $\Delta - JEP$, $\Delta - AP$, Δ-positive Jonsson $(\Delta - PJ)$-theory

INTRODUCTION

The content of this article is related to model theory. More precisely, it examines the model-theoretical properties of positive Jonsson theories. In particular the lattice of positive existential formulae is considered. In the study of complete theories one of the main methods is to use the properties of a topological space $S_n(T)$. In the language of ultrafilters of the Boolean algebra $F_n(T)$, where T is a fixed theory such as classical concepts of model theory as the stability of the model and the theory, the saturation of the model, the homogeneity of the model, diagrams of models have been studied. In the case of positive theory, we can consider the lattice $E_n^+(T)$ of existential formulae, which is a sublattice of the Boolean algebra $F_n^+(T)$. Since the positive existential formulae are not closed in general, with respect to the of Boolean operations, the topological space of positive existential types differs significantly from the positive complete cases. It is clear that such an approach (restriction of $F_n^+(T)$ to $E_n^+(T)$) is a generalization of the case when we deal with the complete theory. Since the positive Jonsson theories are incomplete in general, it would be interesting to consider the properties of the lattice of positive existential formulae in connection with the above mentioned context. The main tool of research for a positive Jonsson theory is a semantic method. The essence of this method is the translation of the properties of the central completion to Jonsson prototype. In this article in addition to the semantic method and other outcomes of Jonsson theories we use notions and results from the work of Volker Weispfenning[1].

The mainstream investigations in model theory belong to western model theory (see p.56 of Ref. 2), but nevertheless we can note a number of papers where similar issues arise in the study of positive Jonsson theories. For example, in Ref. 3 the properties of the class of simple theories of existentially closed models of universal sentences was considered. The theories considered in this article are Robinson theories. And we can see that this class of theories is a natural subclass of the Jonsson theories. For investigating such theories we usually use a semantic method, but even in the case where a $\Delta - PJ$-theory is not a Jonsson theory, the idea of a semantic generalization of the method for the Jonsson theories is useful. The essence of this generalization is the translation of properties of $\Delta - PJ$-central completion to $\Delta - PJ$-prototype. A series of results that establish the relationship between properties of $\Delta - PJ$-theory,

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$\Delta - PJ$-centre of the $\Delta - PJ$-theory and properties of the lattice of equivalence classes of positive existential formulae with respect to these theories is proved. It is essential that the existence of a semantic model of $\Delta - PJ$-theory does not depend on additional axioms of set theory. The results obtained are $\Delta - PJ$-analogues of the results obtained in Refs. 1, 4–6. In Refs. 7, 8 other properties of such lattices are also examined.

We also consider the notion of a Δ-positive Jonsson ($\Delta - PJ$) theory and establish a connection between properties of $\Delta - PJ$-theory, the central completion of the $\Delta - PJ$-theory, and properties of the lattice of equivalence classes of positive existential formulae with respect to this theory. We assume familiarity with definitions and notation on Jonsson theories. All the necessary concepts for Jonsson theories can be found in Refs. 9, 10.

SOME PROPERTIES OF LATTICES OF POSITIVE EXISTENTIAL FORMULAE

We define the notion of Δ-positive Jonsson ($\Delta - PJ$)-theories. Let L be a first-order language. At is a set of atomic formulae of the language. $B^+(At)$ is a set of formulae containing atomic formulae, which is closed with respect to positive Boolean combinations (conjunction and disjunction), subformulae and substitution of variables. $Q(B^+(At))$ is the set of formulae in prenex normal form obtained by the use of quantifiers ($\forall$ and $\exists$) to $B^+(At)$. We call a formula positive if it belongs to $Q(B^+(At))$. A theory is positively axiomatizable if its axioms are positive. $B(L^+)$ is the set of arbitrary Boolean combinations of formulae from L^+. It is easy to see that $\Pi(\Delta) \subseteq B(L^+)$ with $\Delta = B^+(At)$, where $\Pi(\Delta)$ was, as previously described in Refs. 11, 12.

Following Refs. 11, 12 we define Δ-morphisms between structures. Let M and N be two structures of language L $\Delta = B^+(L)$. The mapping $h : M \to N$ is called a Δ-homomorphism (in symbols $h : M \underset{\Delta}{\to} N$), if for any $\varphi(\overline{x}) \in \Delta$, $\forall \overline{a} \in M$ from the fact that $M \models \varphi(\overline{a})$, it follows that $N \models \varphi(h(\overline{a}))$.

Following Refs. 11, 12, the model M is said to begin in N and we say that M continues to N, and that $h(M)$ is a continuation of M. If the map h is injective, we say that h is an immersion of M into N. In what follows we will use the terms Δ-continuation and Δ-immersion. In the frame of this definition (Δ-homomorphism), it is easy to see that an isomorphic embedding, and an elementary embedding are Δ-immersions, when $\Delta = B(At)$ and $\Delta = L$, respectively.

Definition 1 If C is a class of L-structures, then we say that an element M of C is Δ-positively existentially closed in C if every Δ-homomorphism from M to any element of C is a Δ-immersion. The class of all positively existentially Δ-closed models will be denoted by $(E_C^{\Delta})^+$; if $C = \operatorname{Mod} T$ for some theory T, then by E_T, $(E_T^{\Delta})^+$ we mean, respectively, the class of existentially closed and Δ-positively existentially closed models of the theory T.

Definition 2 We say that a theory T admits $\Delta - JEP$, if for any two $A, B \in \operatorname{Mod} T$ there exists $C \in \operatorname{Mod} T$ and Δ-homomorphisms $h_1 : A \underset{\Delta}{\to} C$, $h_2 : B \underset{\Delta}{\to} C$.

Definition 3 We say that a theory T admits $\Delta - AP$, if for any $A, B, C \in \operatorname{Mod} T$ such that $h_1 : A \underset{\Delta}{\to} C$, $g_1 : A \underset{\Delta}{\to} B$, where h_1, g_1 are Δ homomorphisms, there exists $D \in \operatorname{Mod} T$ and $h_2 : C \underset{\Delta}{\to} D$, $g_2 : B \underset{\Delta}{\to} D$, where h_2, g_2 are Δ-homomorphisms such that $h_2 \circ h_1 = g_2 \circ g_1$.

Definition 4 The theory T is called Δ-positive Jonsson ($\Delta - PJ$)-theory if it satisfies the following conditions. (1) T has an infinite model; (2) T is positively $\forall\exists$-axiomatizable; (3) T admits $\Delta - JEP$; (4) T admits $\Delta - AP$.

When $\Delta = B(At)$ we obtain the usual Jonsson theory, the only difference is that it has the only positive $\forall\exists$-axiom.

In what follows, all definitions related to Jonsson theories (in the usual sense) are assumed to be known and can be retrieved, for example, in Ref. 12. Let T be a $\Delta - PJ$-theory. Let $E_n(T)$ be the distributive lattice of existential formulae with n free variables in T. Let $PE_n(T) = \{\varphi \in E_n(T)/\varphi \in \exists(B^+(At))\}$. In this definition, we consider the formula φ up to equivalence with respect to the theory of T, $\varphi = \{\psi \in E_n(L)/T \vdash \varphi \leftrightarrow \psi\}$. Let φ, $\psi \in PE_n(T)$ and $\varphi \bigcap \psi = 0$, where 0 is the minimum of the lattice $PE_n(T)$. Then ψ is called the complement of φ, if $\varphi \bigcup \psi = 1$, where 1 is the maximum of the lattice $PE_n(T)$; ψ is a weak complement of φ, if for all $\mu \in PE_n(T)(\varphi \bigcup \psi) \bigcap \mu = 0 \Rightarrow \mu = 0$. φ is called weakly complemented, if φ has a weak complement. $PE_n(T)$ is called weakly complemented if every $\varphi \in PE_n(T)$ is weakly complemented.

Theorem 1 (Ref. 9) *Let T be a complete for $\exists$-sentences Jonsson theory. Then the following conditions are equivalent.*
(i) *T is perfect;*
(ii) *T^* is model-complete;*
(iii) *$E_n(T)$ is a Boolean algebra.*

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The completeness of the theory for $\exists$-the sentences means that any two models of this theory satisfy the same existential sentences.

We can refine some results of Ref. 1 and of classical model theory to the frame of Jonsson theories.

Theorem 2 (Ref. 13)
(i) *The theory T is model complete if and only if every formula is preserved with respect to the submodels.*
(ii) *The theory T is model complete if and only if every formula is preserved under extensions of models.*

Theorem 3 (Ref. 1) *The theory T positively model-complete if and only if each $\varphi^T \in E_n(T)$ has a positive existential complement.*

The following result connects completeness and model completeness in the frame of Jonsson theories. It is studies properties of companions of the Jonsson theories. The following result, that the property of model completeness and the property of completeness coincides for any Jonsson theory has a relation to a well-known theorem of P. Lindstrom concerning this notions Ref. 2.

Theorem 4 (Ref. 9) *Let T be a perfect Jonsson theory. The following conditions are equivalent.*
(i) *T is complete.*
(ii) *T is model complete.*

Theorem 5 (Ref. 1) *Existential formulae φ is invariant in $\mathrm{Mod}(Th_{\forall\exists}(E_T))$, where $E(T)$ is the class of existentially closed models of T, if and only if φ^T is weakly complemented in $E(T)$.*

We introduce the necessary definitions and state some known results which establish the relationship between model completeness, quantifier elimination, model completeness of the positive properties of the lattice theory and existential formulae $E_n(T)$.

Theorem 6 (Ref. 2)
(i) *Let T' be a model companion of the theory T, where T is a universal theory. Then, T' is a model completion of T, if and only if the theory T admits elimination of quantifiers.*
(ii) *Let T' be a model companion of T. Then, T' is a model completion of T, if and only if the theory T has the amalgamation property.*

Theorem 7 (Ref. 1) *The theory T has a model completion if and only if $E_n(T)$ is a Stone algebra.*

Theorem 8 (Ref. 1) *The theory $T_\forall$ has a model completion if and only if each $\varphi^T \in E_n(T)$ has a weak quantifier-free complement.*

Let us note that since a theory which is complete for existential sentences satisfies the joint embedding property, but the converse is not true, we see that the condition of existential-completeness in our theorems cannot be eliminated. Hereinafter all considered theories will be complete for the Σ^+-sentences. We can play with Δ. For example, from Ref. 14 one can assume that Δ be equals to the minimal fragment $\Delta = B^+(At)$. Let $\Delta = B^+(At)$

Theorem 9 *Let T be a $\Delta - PJ$ theory, T^*_Δ be the centre of the theory T. Then*
(i) *T^*_Δ admits elimination of quantifiers if and only if every $\varphi \in PE_n(T)$ has quantifier-free complement;*
(ii) *$T^*_\Delta \Delta is PJ$-positively model-complete if and only if every $\varphi \in PE_n(T)$ has a positive existential complement.*

Proof: We need to consider two cases.

(a) T is a Jonsson theory. We prove point (i). Since T is a Jonsson theory, then as a centre, we consider the $T^*_\Delta = Th(C)$, and one admits elimination of quantifiers. From this it follows that T^*_Δ is submodel complete. Then the theory T^*_Δ by definition is model complete, and so by Theorem 1 $E_n(T)$ is a Boolean algebra, i.e., every $\varphi \in PE_n(T)$ has a complement. Since $PE_n(T) \subset E_n(T)$, then every $\varphi \in PE_n(T)$ has a complement. By elimination of quantifiers T^*_Δ, since T^*_Δ is the completion of the theory T, then with respect to the theory T each $\varphi \in PE_n(T)$ has a quantifier-free complement.

Conversely, suppose that each $\varphi \in PE_n(T)$ has a quantifier-free complement. Then $PE_n(T)$ is a Boolean algebra, then by Theorem 1, T^*_Δ is model-complete, and then, in turn, by virtue of Theorem 2(ii), we have that any formula with respect to T^*_Δ is equivalent to some existential formula, i.e., this formula belongs to the class $PE_n(T^*_\Delta)$. By Σ^+-completeness of the theory T we have that $PE_n(T) = PE_n(T^*_\Delta)$. Consequently, by virtue of the fact that each $\varphi \in PE_n(T)$ has quantifier-free complement, and $PE_n(T)$ is a Boolean algebra, every formula in $PE_n(T^*_\Delta)$ is without quantifiers. Therefore, the theory of T^*_Δ admits elimination of quantifiers.

We now prove (ii). Let the theory of T^*_Δ be $\Delta - PJ$-positively model-complete. Then by the definition of $\Delta - PJ$-positive model completeness of T^*_Δ, it is $\Delta - PJ$-model-complete and for every existential formula φ there is a positive existential formula of ψ

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such that $T^*_\Delta| - .\varphi \leftrightarrow \psi$. By Theorem 1, $PE_n(T)$ is a Boolean algebra, i.e., every $\varphi \in PE_n(T)$ has an existential complement, and as for every existential formula φ there is a positive existential formula of ψ such that $T^*_\Delta| - .\varphi \leftrightarrow \psi$, we find that each $\varphi \in PE_n(T)$ has a positive existential complement.

We prove the sufficiency of (ii). Let every $\varphi \in PE_n(T)$ have a positive existential complement. Then, by Theorem 3, the theory T is positively model-complete in the sense of Ref. 15 (recall the definition of A. Macintyre: theory T is positively model-complete if T is model-complete and every existential L-formula is equivalent in T to some positive existential L-formula). Then, by Theorem 4 we have that the theory T is complete, and as the theory T^*_Δ is the centre of the theory T, we see that $T = T^*_\Delta$. Thus T^*_Δ is positively model-complete in the sense of Ref. 15, but since $\Delta = B^+(At)$, and hence, $B^+(At) \subset B(At)$, i.e., T^*_Δ is $\Delta - PJ$-positively model-complete. The proof of case (a) is completed.

Case (b) T is not a Jonsson theory. We prove (i). Since T is not a Jonsson theory, then as $\operatorname{Mod} T$ we consider E^+_T, and the centre of T is the theory $T^*_\Delta = Th(U)$, where U is the universal domain of the language L, which is a model of the theory T. $T^*_\Delta = Th(U)$ and it admits elimination of quantifiers. From that, it follows that T^*_Δ is submodel complete. Then, by Theorem 3, it follows that each $\varphi \in E_n(T^*_\Delta)$ has a quantifier-free complement. Due to the fact that we work in E^+_T, $E_n(T^*_\Delta) = PE_n(T^*_\Delta)$. By Σ^+-completeness of the theory T, $PE_n(T^*_\Delta) = PE_n(T)$, and hence every $\varphi \in PE_n(T)$ has a quantifier-free complement.

Conversely, suppose that each $\varphi \in PE_n(T)$ has a quantifier-free complement. Then $PE_n(T)$ is a Boolean algebra. By Σ^+-completeness of the theory T, we have that $PE_n(T) = PE_n(T^*_\Delta)$. Consequently, by virtue of the fact that each $\varphi \in PE_n(T^*_\Delta)$ has a quantifier-free complement, then by Theorem 3, it follows that T^*_Δ is submodel complete. Consequently, the theory T^*_Δ admits elimination of quantifiers.

We now prove (ii). Let the theory T^*_Δ is $\Delta - PJ$-positively model-complete. Then by the definition of $\Delta - PJ$-positive model completeness of theory, T^*_Δ is $\Delta - PJ$-model-complete and for every existential formula φ there is a positive existential formula ψ such that true $T^*_\Delta \vdash \varphi \leftrightarrow \psi$. But since $PE_n(T) \subset PE_n(T^*_\Delta)$, and $PE_n(T^*_\Delta)$ is a Boolean algebra, i.e., every $\varphi \in PE_n(T^*_\Delta)$ has an existential complement, and as for every existential formula φ there is a positive existential formula ψ such that $T^*_\Delta \vdash \varphi \leftrightarrow \psi$, we can conclude that each $\varphi \in PE_n(T^*_\Delta)$ has a positive existential complement. But then every $\varphi \in PE_n(T)$ has a positive existential complement. Thus the necessity of (ii) is proved.

We prove the sufficiency of (ii). Let every $\varphi \in PE_n(T)$ have a positive existential complement. Then, by Theorem 3, the theory T is positively model-complete in the sense of Ref. 15, and therefore, by definition is model complete. But since $\operatorname{Mod} T = E^+_T$, and for any A of E^+_T, A is immersed in U, it follows that U is saturated in its power for positive Δ-types. Thus the theory T is $\Delta - PJ$-perfect. Then $T^*_\Delta \Delta - PJ$ is positively model-complete. □

Theorem 10 *Let T be a $\Delta - PJ$-theory. Then the following conditions are equivalent.*
(i) *T is $\Delta - PJ$-perfect;*
(ii) *$PE_n(T)$ is weakly complemented ;*
(iii) *$PE_n(T)$ is a Stone lattice.*

Proof: We consider two cases.

(a) T is a Jonsson theory. We will prove that (i) implies (ii). Let the Jonsson theory T be $\Delta - PJ$-perfect. Then T is perfect in the sense of the Jonsson theory. Then the theory T has a model companion T^M. From Ref. 11 it is known that in this case in such frame of conditions of our statement we have that $T^M = T^0$, where $T^0 = Th_{\forall\exists}(E_T)$, Kaiser's hull of the Jonsson theory T. Since by definition the model companion T^M is model-complete, we have therefore that every formula of the language is persistent with respect to the submodels $\operatorname{Mod} T^M$. Consequently, every existential formula of the language is persistent with respect to the submodels $\operatorname{Mod} T^M$, while at the same time every existential formula of the language is persistent under extensions of models in the $\operatorname{Mod} T^M$, and therefore, this formula is invariant in $\operatorname{Mod} T^M$. Hence, by Theorem 5, it follows that every existential formula is weakly complemented. Thus $E_n(T)$ is weakly complemented. Therefore, since $PE_n(T) \subset E_n(T)$,we have that $PE_n(T)$ is weakly complemented.

We will prove that (ii) implies (i). If $PE_n(T)$ is weakly complemented, then the theory T has a positive model companion. Then the theory T is perfect, and its positive model companion is T^*_Δ. Therefore, the theory T^*_Δ is positively model-complete. Therefore, $T\Delta - PJ$.

We will prove that (i) implies (iii). Let the T-$\Delta - PJ$ be perfect. Then the theory T is perfect in the sense of the Jonsson theory and the theory T has a model companion. Make a note that by point 2 of Theorem 6 the model companion of the Jonsson theory is its model completion. Then, by Theorem 7, it

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follows that $E_n(T)$ is a Stone lattice. Therefore, since $PE_n(T) \subset E_n(T)$, we have that $PE_n(T)$ is a Stone lattice.

We will prove that (iii) implies (i). If $PE_n(T)$ is a Stone lattice, then by Theorem 7, the theory T has a model companion, and, consequently, the theory T is perfect. And hence, the theory T is $\Delta - PJ$-perfect.

(b) T is not a Jonsson theory. We will prove that (i) implies (ii). Let the theory T is $\Delta - PJ$-perfect. Since T is not a Jonsson theory, then as $\operatorname{Mod} T$ we consider E_T^+, and the centre of T is the theory of $T_\Delta^* = Th(U)$, where U is a k-universal domain of the language L, which is a model of the theory of T. The theory of T_Δ^* is $\Delta - PJ$ is positive model complete. Then T_Δ^* L model-complete, and every formula of the language is persistent with respect to the submodels in $\operatorname{Mod} T_\Delta^*$. Consequently, every existential formula of the language is persistent with respect to the submodels in $\operatorname{Mod} T_\Delta^*$. At the same time, every existential formula of the language is persistent under extensions of models in the $\operatorname{Mod} T_\Delta^*$, and therefore, by definition, this formula is invariant in $\operatorname{Mod} T_\Delta^*$. Hence, by Theorem 5, it follows that every existential formula is weakly complemented. Thus $E_n(T)$ is weakly complemented. Therefore, since $PE_n(T) \subset E_n(T)$, $PE_n(T)$ is weakly complemented.

We will prove that (ii) implies (i). If $PE_n(T)$ is weakly complemented, then the theory T has a positive model companion. And then every $\varphi \in PE_n(T)$ has a positive existential weak complement. Since $PE_n(T) \subset E_n(T)$, then every $\varphi \in PE_n(T)$ has a positive existential complement. Then the theory T is positively model-complete in the sense of Ref. 14, and therefore, by definition is model complete. But since $\operatorname{Mod} T = E_T^+$, and for any A from E_T^+, A is immersed into the U, it follows that U is saturated in its power for positive Δ-types. Thus the theory T is $\Delta - PJ$-perfect.

We will prove that (ii) implies (iii). Since $PE_n(T)$ is weakly complemented, then the theory T has a model companion which we denote by T_+^M. By Theorem 10, T_+^M is a model completion. Then $PE_n(T)$ is a Stone lattice.

We will prove that (iii) implies (ii). Let $PE_n(T)$ be a Stone lattice. Since $PE_n(T)$ is a Stone lattice, then the theory T has a model companion T^M. Since $\operatorname{Mod} T = E_T^+$, then $T^M = T_+^M$. Then it follows that $T_+^M = T_+^0 = Th_{\forall\exists}(E_T^+)$. Hence, since $U \in E_T^+$, it follows that $T-$ is $\Delta - PJ$-perfect. □

Lemma 1 *Ref. 2. If T has a model companion T^M, then $T_\forall$ has a model companion $(T_\forall)^M$ and $T^M = (T_\forall)^M$.*

Theorem 11 *Let T be a $\Delta - PJ$-theory. Then the following conditions are equivalent.*
(i) *T_Δ^* is a $\Delta - PJ$-theory;*
(ii) *each $\varphi \in PE_n(T)$ has a weak quantifier-free complement.*

Proof: We consider two cases.

(a) T is a Jonsson theory. We will prove that (i) implies (ii). Since T is a Jonsson theory, then as a centre, we consider the theory $T_\Delta^* = Th(C)$, and T_Δ^*-$\Delta - PJ$-theory. If T_Δ^* is a Jonsson theory, it follows from Ref. 9 that the theory of T is perfect. Then the theory of T has a model companion, equal to the theory of T_Δ^*, which is the model completion of the theory of T. By virtue of the mutual model consistency of the theory T and the theory of $T_\forall$ (all universal consequences of the theory of T) and Lemma 1, we have that the central completion of the theory T is a model completion of the theory $T_\forall$. Then every $\varphi^T \in E_n(T)$ has a weak quantifier-free complement. Since $PE_n(T) \subset E_n(T)$, then every $\varphi \in PE_n(T)$ has a weak quantifier-free complement.

We will prove that (ii) implies (i). Every $\varphi \in PE_n(T)$ has a weak quantifier-free complement. Then every $\varphi \in PE_n(T)$ has a weak complement, that is, $PE_n(T)$ is weakly complemented. Then, by Theorem 2, the theory T is $\Delta - PJ$ - perfect.

(b) T is not a Jonsson theory. We will prove that (i) implies (ii). Let T_Δ^* be a $\Delta - PJ$-theory, in this case we consider as above that $E_{T_\Delta^*}^+ = \operatorname{Mod} T_\Delta^*$. One can see that $E_{T_\Delta^*}^+$ is contained in $\operatorname{Mod} T$. But we know that the positive Kaiser's hull $T_+^0 = (E_T^+)$ is always a $\Delta - PJ$ - theory, and for our fixed $\Delta = B^+(At)$ due to the fact that T_Δ^* is $\Delta - PJ$-theory, it follows that $\operatorname{Mod} T_+^0 = E_T^+ = \operatorname{Mod} T$. This means that the theory T is $\Delta - PJ$-perfect. Then the theory T has a model completion. Then $T_\forall$ has a model completion. Hence, by Theorem 8, every $\varphi \in E_n(T)$ has a weak quantifier-free complement. Then every $\varphi \in PE_n(T)$ has a weak quantifier-free complement.

We will prove that (ii) implies (i). Let every $\varphi \in PE_n(T)$ have a weak quantifier-free complement. Then every $\varphi \in PE_n(T)$ has a weak complement, that is, $PE_n(T)$ is weakly complemented. Then, by Theorem 2, the theory T is $\Delta - PJ$-perfect. This means that T_Δ^* is a $\Delta - PJ$-theory. □

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