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The automorphism group of Poisson algebras on $k\,[x,y]$

Poisson algebras play a key role in the Hamiltonian mechanics, symplectic geometry and also are central in the study of quantum groups. At present, Poisson algebras are investigated by the many mathematicians of Russia, France, the USA, Brazil, Argentina, Bulgaria etc. The purpose of the present paper is to describe the automorphism groups of polynomial algebras endowed with additional structure, namely, with Poisson brackets. For any $f \in k\,[x,y]$ one can transform associative-commutative algebra $k\,[x,y]$ into a Poisson algebra P_f by defining a Poisson bracket by the rule $\{x,y\} = f$. Obviously, a structure of the automorphism group G_f of Poisson algebra P_f depends on the element f. A complete description of group G_f is given for the polynomial f of rank less or equals to 1. In present paper all algebras are considered over any field k of characteristic 0.

Keywords: Poisson algebras, polynomial algebras, automorphisms, additional structure.

Introduction

It is known [1–4] that the automorphisms of polynomial algebras $k\,[x,y]$ and free associative algebras $k\,\langle x,y\rangle$ in two variables are products of affine automorphisms

$$\varphi = (\alpha_{11}x + \alpha_{21}y + \beta_1, \alpha_{12}x + \alpha_{22}y + \beta_2)\,, \alpha_{ij}, \beta_j \in k,$$

and triangular automorphisms

$$\psi = (\alpha_1 x + f\,(y)\,, \alpha_2 y + \beta_2)\,, \alpha_1, \alpha_2 \in k^*, f\,(y) \in k\,[y]\,, \beta_2 \in k$$

i.e are *tame*.

In work [5] is proved that the automorphisms of two-generated free Poisson algebras $k\,\{x,y\}$ over a field k of characteristic 0 are time. Moreover [1, 4, 5], the automorphism groups of algebras $k\,[x,y]$, $k\,\langle x,y\rangle$, $k\,\{x,y\}$ are isomorphic, i.e.

$$Aut\,k[x_1,x_2] \cong Aut\,k < x_1, x_2 > \cong Aut\,k\{x_1,x_2\}.$$

The purpose of the present paper is to describe the automorphism groups of polynomial algebras endowed with additional structure, namely, with Poisson brackets. For any $f \in k\,[x,y]$ one can transform associative-commutative algebra $k\,[x,y]$ into a Poisson algebra P_f by defining a Poisson bracket by the rule $\{x,y\} = f$. Obviously, a structure of the automorphism group G_f of Poisson algebra P_f depends on the element f. A complete description of group G_f is given for the polynomial f of rank less or equals to 1.

In section 2 we introduce the basic definitions, examples of Poisson algebras and collect the informations necessary for the further work. In section 3 we study the automorphism group of Poisson algebra P_f.

In present paper all algebras are considered over any field k of characteristic 0.

Poisson brackets on $k\,[x,y]$

Recall that a vector space P over a field K endowed with two bilinear operations $x \cdot y$ (a multiplication) and $\{x,y\}$ (a Poisson bracket) is called *a Poisson algebra* if P is a commutative associative algebra under $x \cdot y$, P is a Lie algebra under $\{x,y\}$, and P satisfies the following identity

$$\{x, y \cdot z\} = \{x,y\} \cdot z + y \cdot \{x,z\}.$$

There are two important classes of Poisson algebras.

1. Symplectic algebras S_n. For each n the algebra S_n is a polynomial algebra $k[x_1, y_1, \ldots, x_n, y_n]$, endowed with the Poisson bracket defined by $\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0$, where δ_{ij} is the Kronecker symbol and $1 \leq i, j \leq n$.

2. Symmetric Poisson algebras $PS(L)$. Let L be a Lie algebra with a linear basis $e_1, e_2, \ldots, e_k, \ldots$. Then $PS(L)$ is the usual polynomial algebra $K[e_1, e_2, \ldots, e_k, \ldots]$ endowed with the Poisson bracket defined by $\{e_i, e_j\} = [e_i, e_j]$ for all i, j, where $[x, y]$ is the multiplication of the Lie algebra L.

Let's consider the algebra $k\left[x, y\right]$. Let $f \in k\left[x, y\right]$. For any $a, b \in k\left[x, y\right]$ we put

$$\{a, b\} = \left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}\right) f. \tag{1}$$

Lemma 1. The bracket $\{\cdot, \cdot\}$ sets up the structure of Poisson algebra on polynomial algebra $k\left[x, y\right]$.

Proof. For any $a, b, c \in k\left[x, y\right]$ it's enough to verify the Leibniz identity and Jacobi identity:

$$\{a \cdot b, c\} = \{a, c\} b + a \{b, c\},$$

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0.$$

Initially we verify the implementation of the Leibniz identity. Using a formula (1) we have

$$\{a \cdot b, c\} = \left(\frac{\partial (ab)}{\partial x}\frac{\partial c}{\partial y} - \frac{\partial c}{\partial x}\frac{\partial (ab)}{\partial y}\right) f = \left(\left(\frac{\partial a}{\partial x}b + a\frac{\partial b}{\partial x}\right)\frac{\partial c}{\partial y} - \frac{\partial c}{\partial x}\left(\frac{\partial a}{\partial y}b + a\frac{\partial b}{\partial y}\right)\right) f =$$

$$= \left(\frac{\partial a}{\partial x}\frac{\partial c}{\partial y}b + a\frac{\partial b}{\partial x}\frac{\partial c}{\partial y} - \frac{\partial c}{\partial x}\frac{\partial a}{\partial y}b - a\frac{\partial c}{\partial x}\frac{\partial b}{\partial y}\right) f =$$

$$= \left(\frac{\partial a}{\partial x}\frac{\partial c}{\partial y} - \frac{\partial c}{\partial x}\frac{\partial a}{\partial y}\right) fb + a\left(\frac{\partial b}{\partial x}\frac{\partial c}{\partial y} - \frac{\partial c}{\partial x}\frac{\partial b}{\partial y}\right) f = \{a, c\} b + a \{b, c\}.$$

Now we verify the implementation of the Jacobi identity. If $a \in k$ then the equation is obvious. If $a = x$ then using a formula (1) we get

$$\{\{x, b\}, c\} + \{\{b, c\}, x\} + \{\{c, x\}, b\} = \left\{\frac{\partial b}{\partial y}f, c\right\} + \left\{\left(\frac{\partial b}{\partial x}\frac{\partial c}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial c}{\partial x}\right) f, x\right\} -$$

$$- \left\{\frac{\partial c}{\partial y}f, b\right\} = \left\{\frac{\partial b}{\partial y}f, c\right\} + \left\{\frac{\partial b}{\partial x}\frac{\partial c}{\partial y}f, x\right\} - \left\{\frac{\partial b}{\partial y}\frac{\partial c}{\partial x}f, x\right\} - \left\{\frac{\partial c}{\partial y}f, b\right\} =$$

$$= \left\{\frac{\partial b}{\partial y}f, c\right\} + \left\{\frac{\partial b}{\partial x}, x\right\}\frac{\partial c}{\partial y}f + \frac{\partial b}{\partial x}\left\{\frac{\partial c}{\partial y}f, x\right\} - \left\{\frac{\partial b}{\partial y}, x\right\}\frac{\partial c}{\partial x}f - \frac{\partial b}{\partial y}\left\{\frac{\partial c}{\partial x}f, x\right\} -$$

$$- \left\{\frac{\partial c}{\partial y}f, b\right\} = \left(\frac{\partial}{\partial x}\left(\frac{\partial b}{\partial y}f\right)\frac{\partial c}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial b}{\partial y}f\right)\frac{\partial c}{\partial x}\right) f - \frac{\partial^2 b}{\partial x \partial y}\frac{\partial c}{\partial y}f^2 - \frac{\partial b}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y}f\right) f +$$

$$+ \frac{\partial^2 b}{\partial y^2}\frac{\partial c}{\partial x}f^2 + \frac{\partial b}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial c}{\partial x}f\right) f - \left(\frac{\partial}{\partial x}\left(\frac{\partial c}{\partial y}f\right)\frac{\partial b}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y}f\right)\frac{\partial b}{\partial x}\right) f =$$

$$= \frac{\partial^2 b}{\partial x \partial y}\frac{\partial c}{\partial y}f^2 + \frac{\partial f}{\partial x}\frac{\partial b}{\partial y}\frac{\partial c}{\partial y}f - \frac{\partial c}{\partial x}\frac{\partial^2 b}{\partial y^2}f^2 - \frac{\partial c}{\partial x}\frac{\partial f}{\partial y}\frac{\partial b}{\partial y}f - \frac{\partial^2 b}{\partial x \partial y}\frac{\partial c}{\partial y}f^2 -$$

$$- \frac{\partial b}{\partial x}\frac{\partial^2 c}{\partial y^2}f^2 - \frac{\partial b}{\partial x}\frac{\partial f}{\partial y}\frac{\partial c}{\partial y}f + \frac{\partial^2 b}{\partial y^2}\frac{\partial c}{\partial x}f^2 + \frac{\partial b}{\partial y}\frac{\partial^2 c}{\partial x \partial y}f^2 + \frac{\partial b}{\partial y}\frac{\partial f}{\partial y}\frac{\partial c}{\partial x}f -$$

$$- \frac{\partial^2 c}{\partial x \partial y}\frac{\partial b}{\partial y}f^2 - \frac{\partial f}{\partial x}\frac{\partial c}{\partial y}\frac{\partial b}{\partial y}f + \frac{\partial^2 c}{\partial y^2}\frac{\partial b}{\partial x}f^2 + \frac{\partial f}{\partial y}\frac{\partial c}{\partial y}\frac{\partial b}{\partial x}f = 0.$$

Suppose that $deg\left(a\right) \geq 2$. Then can be considered that $a = a^{'} \cdot x$, $deg\left(a^{'}\right) < deg\left(a\right)$. We have

$$\left\{\left\{a^{'} \cdot x, b\right\}, c\right\} + \left\{\{b, c\}, a^{'} \cdot x\right\} + \left\{\left\{c, a^{'} \cdot x\right\}, b\right\} = \left\{\left\{a^{'}, b\right\} x + a^{'}\{x, b\}, c\right\} +$$

$$+\left\{\{b,c\},a'\right\}x+a'\{\{b,c\},x\}+\left\{\left\{c,a'\right\}x+a'\{c,x\},b\right\}=\left\{\left\{a',b\right\}x,c\right\}+$$

$$+\left\{a'\{x,b\},c\right\}+\left\{\{b,c\},a'\right\}x+a'\{\{b,c\},x\}+\left\{\left\{c,a'\right\}x,b\right\}+\left\{a'\{c,x\},b\right\}=$$

$$=\left\{\left\{a',b\right\},c\right\}x+\left\{a',b\right\}\{x,c\}+\left\{a',c\right\}\{x,b\}+a'\{\{x,b\},c\}+\left\{\{b,c\},a'\right\}x+$$

$$+a'\{\{b,c\},x\}+\left\{\left\{c,a'\right\},b\right\}x+\left\{c,a'\right\}\{x,b\}+\left\{a',b\right\}\{c,x\}+a'\{\{c,x\},b\}=$$

$$=\left(\left\{\left\{a',b\right\},c\right\}+\left\{\{b,c\},a'\right\}+\left\{\left\{c,a'\right\},b\right\}\right)x+$$

$$+a'\left(\{\{x,b\},c\}+\{\{b,c\},x\}+\{\{c,x\},b\}\right)=0.\ \square$$

Poisson algebra on $k[x,y]$ given by the Poisson bracket (1) we denote by P_f. Note that

$$\{x,y\}=f. \tag{2}$$

The automorphism group

The element g of polynomial algebra $k[x_1,x_2,...,x_n]$ has the *rank* r (see the definition, for example in [6]) if the next two conditions are implemented:

1) there exists the automorphism φ of the algebra $k[x_1,x_2,...,x_n]$ such that $\varphi(g)\in k[x_1,x_2,...,x_r]$;

2) $\varphi(g)\notin k[x_1,x_2,...,x_{r-1}]$ not for any automorphism φ of the algebra $k[x_1,x_2,...,x_n]$.

The rank of the element g we denote by $rank(g)$. If $f\in k[x,y]$ then $rank(f)$ might receive the values 0, 1, 2. The rank of the algebra P_f is called the number $rank(f)$.

Note that $rank(f)=0$ if and only if $f\in k$. 2 cases are possible:

1) $f=0$; 2) $f\neq 0$.

If $f=0$ then Poisson bracket is zero. Therefore

$$Aut_kP_0\cong Aut_kk[x,y].$$

If $0\neq f=\alpha\in k$ then having exchanged the variables $x'=\alpha^{-1}x$, $y'=y$ can be considered that

$$\{x,y\}=1,$$

i.e. $P_\alpha\cong P_1$. Note that P_1 is symplectic algebra, i.e. isomorphic to the algebra S_1.

If φ – automorphism of algebra $k[x,y]$ then

$$\{\varphi(x),\varphi(y)\}=J(\varphi)\cdot\{x,y\},$$

where

$$J(\varphi)=\begin{vmatrix}\frac{\partial\varphi(x)}{\partial x} & \frac{\partial\varphi(x)}{\partial y}\\ \frac{\partial\varphi(y)}{\partial x} & \frac{\partial\varphi(y)}{\partial y}\end{vmatrix}$$

– Jacobian φ.

Therefore $\varphi\in Aut_kP_1$ if and only if

$$J(\varphi)=1.$$

Thus Aut_kP_1 consists subgroup $Aut_kk[x,y]$ with Jacobian 1. It is known [7] that such group isomorphic to the automorphism group of Weyl algebra W_1, i.e.

$$Aut_kP_1\cong Aut_kW_1.$$

Recall that Weyl algebra W_1 is associative algebra (with unit 1) with generators x,y and defining relation $[x,y]=1$.

Suppose that $rank(f)=1$. Then, by the rank definition, there exists the automorphism φ of the algebra $k[x,y]$ such that $\varphi(f)=g(x)\notin k$. Let's denote $x'=\varphi^{-1}(x)$, $y'=\varphi^{-1}(y)$. Therefore

$$\left\{x',y'\right\}=J\left(\varphi^{-1}\right)\cdot\{x,y\}=\gamma f(x,y)=\gamma\varphi^{-1}(g(x))=\gamma g\left(\varphi^{-1}(x)\right)=\gamma g\left(x'\right)=g'\left(x'\right).$$

Thus $\left\{x^{'}, y^{'}\right\} = g^{'}\left(x^{'}\right)$. Having exchanged the variables $x^{'} = x$, $y^{'} = y$, can be considered that $\{x, y\} = = f(x) \notin k$.

Further we fix the polynomial $f \in k[x]$, $n = deg(f) \geq 1$.

Lemma 2. The map

$$\sigma_{\alpha,\beta,\gamma}: \begin{array}{l} x \to \alpha x + \beta \\ y \to \gamma y \end{array}$$

is automorphism of the algebra P_f if and only if the next conditions are implemented:

1) $\alpha \in k^*$, $\gamma = \alpha^{n-1}$;

2) *the set of roots of the polynomial $f(x)$ is invariant concerning the affine map*

$$\varphi_{\alpha,\beta}: z \mapsto \frac{z-\beta}{\alpha}$$

of the space $\bar{k}$, where $\bar{k}$ – algebraic closure of the field k.

Proof. The map $\sigma_{\alpha,\beta,\gamma}$ sets up the automorphism of algebra P_f if and only if $\sigma_{\alpha,\beta,\gamma}$ is the automorphism of polynomial algebra $k[x, y]$ and $\sigma_{\alpha,\beta,\gamma}$ retains the unique relation (2), i.e.

$$\{\sigma_{\alpha,\beta,\gamma}(x), \sigma_{\alpha,\beta,\gamma}(y)\} = \sigma_{\alpha,\beta,\gamma}(f).$$

Thus the map $\sigma_{\alpha,\beta,\gamma}$ is the automorphism if and only if $\alpha, \gamma \in k^*$ and

$$\alpha\gamma f(x) = f(\alpha x + \beta). \tag{3}$$

Let

$$f(x) = a_0 + a_1 x + ... + a_n x^n, \quad a_n \neq 0.$$

Using the ratio (3) we get

$$\alpha\gamma\left(a_0 + a_1 x + ... + a_n x^n\right) = a_0 + a_1(\alpha x + \beta) + ... + a_n(\alpha x + \beta)^n.$$

Comparing the leading coefficients, from here we get $\alpha\gamma a_n = a_n\alpha^n$, i.e. $\gamma = \alpha^{n-1}$.

Let

$$f(x) = a_n(x - \alpha_1)(x - \alpha_2) \cdot ... \cdot (x - \alpha_n),$$

where $\alpha_1, \alpha_2, ..., \alpha_n \in \bar{k}$ – the roots of polynomial $f(x)$.

Then from the equation (3) follows that

$$a_n\alpha^n(x - \alpha_1)(x - \alpha_2) \cdot ... \cdot (x - \alpha_n) =$$

$$= a_n(\alpha x + \beta - \alpha_1)(\alpha x + \beta - \alpha_2) \cdot ... \cdot (\alpha x + \beta - \alpha_n) =$$

$$= a_n\alpha^n\left(x - \frac{\alpha_1 - \beta}{\alpha}\right)\left(x - \frac{\alpha_2 - \beta}{\alpha}\right) \cdot ... \cdot \left(x - \frac{\alpha_n - \beta}{\alpha}\right).$$

Therefore the affine map

$$\varphi_{\alpha,\beta}: z \mapsto \frac{z-\beta}{\alpha}$$

of the space $\bar{k}$ leaves the set of roots of the polynomial $f(x)$ is invariant. □

Corollary 1. If the map $\sigma_{\alpha,\beta,\gamma}$ from the formulation of Lemma 2 is the automorphism of the algebra P_f then β, γ are uniquely determined by $\alpha \in k^*$.

Proof. Lemma 2 gives $\gamma = \alpha^{n-1}$. Comparing the coefficients at x^{n-1} from the equation (3) we get

$$\alpha\gamma a_{n-1} = \alpha^{n-1}a_{n-1} + n\alpha^{n-1}\beta a_n,$$

i.e.

$$\beta = \frac{(\alpha - 1)a_{n-1}}{na_n}. \ \square$$

Since the automorphism $\sigma_{\alpha,\beta,\gamma}$ from Lemma 2 is uniquely determined by α then this automorphism further we denote by σ_{α}. By H we denote a subgroup of the group G_f, which consists all automorphisms σ_{α}, where $\alpha \in k^*$.

Lemma 3.

a) If the polynomial $f(x)$ has at least two different roots then the group H is finite.

b) If the polynomial $f(x)$ has a unique root then $H \cong k^*$.

Proof. By Lemma 2 the mapping $\varphi_{\alpha,\beta}$ commutes the roots of polynomial $f(x)$. Since $f(x)$ has no more than n different roots then there exists m such that $m \leq n$ and $\varphi^m_{\alpha,\beta}$ leaves all roots of polynomial $f(x)$ immobile. Direct calculations give

$$\varphi^m_{\alpha,\beta}(z) = \frac{z - \beta\left(1 + \alpha + ... + \alpha^{m-1}\right)}{\alpha^m}.$$

If the polynomial $f(x)$ has at least two different roots then from here follows $\varphi^m_{\alpha,\beta} = id$. Therefore $\alpha^m = 1$. Thus α might receive a finite set of values, i.e. the group H is finite.

Suppose that the polynomial $f(x)$ has a unique root α_1. Then

$$f(x) = a_n (x - \alpha_1)^n .$$

By Viete formula we have

$$n\alpha_1 = -\frac{a_{n-1}}{a_n}.$$

Since the field k has a characteristic 0 then we get $\alpha_1 \in k$.

Show that

$$\alpha \in k^* \to \sigma_\alpha \in H$$

is isomorphism. It is sufficient to show that the equality $\sigma_{\alpha\alpha'} = \sigma_\alpha \sigma_{\alpha'}$ is implemented. Considering Corollary 1 we have

$$\sigma_{\alpha\alpha'}(x) = \alpha\alpha' x + \frac{\left(\alpha\alpha' - 1\right) a_{n-1}}{na_n},$$

$$\sigma_\alpha \sigma_{\alpha'}(x) = \sigma_\alpha\left(\alpha' x + \frac{\left(\alpha' - 1\right) a_{n-1}}{na_n}\right) = \alpha\alpha' x + \frac{\alpha\left(\alpha' - 1\right) a_{n-1}}{na_n} + \frac{(\alpha - 1)\, a_{n-1}}{na_n} =$$

$$= \alpha\alpha' x + \frac{\left(\alpha\alpha' - 1\right) a_{n-1}}{na_n},$$

i.e. $\sigma_{\alpha\alpha'}(x) = \sigma_\alpha \sigma_{\alpha'}(x)$. Also

$$\sigma_{\alpha\alpha'}(y) = (\alpha\alpha')^{n-1} y = \alpha^{n-1}\alpha'^{n-1} y,$$

$$\sigma_\alpha \sigma_{\alpha'}(y) = \sigma_\alpha\left(\alpha'^{n-1} y\right) = \alpha^{n-1}\alpha'^{n-1} y,$$

i.e. $\sigma_{\alpha\alpha'}(y) = \sigma_\alpha \sigma_{\alpha'}(y)$. □

Let's consider the map

$$\tau: \begin{array}{l} x \to x \\ y \to y + h(x), \quad h(x) \in k[x]. \end{array}$$

The map τ is the automorphism of algebra P_f, since τ retains the unique relation (2), i.e.

$$\{\tau(x), \tau(y)\} = \tau(f).$$

We have

$$\{x, y + h(x)\} = f(x),$$

i.e.

$$\{x, y\} = f(x).$$

By T we denote a subgroup of the group G_f, which consists all automorphisms τ.

Theorem 1. The automorphism group G_f of Poisson algebra P_f, where $f = f(x)$ – a polynomial of rank 1, is a semi-direct product of H and T, i.e.

$$G_f \cong H \leftthreetimes T.$$

Proof. Initially we show that the group G_f is generated by automorphisms $\sigma_\alpha \in H$ and $\tau \in T$. Let's consider «commutator» ideal $C(P_f)$ of algebra P_f. Note that $C(P_f) = f(x) \cdot P_f$. The ideal $C(P_f)$ is characteristic, i.e. is invariant concerning all endomorphisms. Therefore for any $\varphi \in G_f$ we have

$$\varphi(f(x)) = f(x) \cdot b(x,y),\ \varphi^{-1}(f(x)) = f(x) \cdot c(x,y).$$

Then

$$f(x) = \varphi^{-1}(\varphi(f(x))) = \varphi^{-1}(f(x) \cdot b(x,y)) = \varphi^{-1}(f(x)) \cdot \varphi^{-1}(b(x,y)) =$$
$$= f(x) \cdot c(x,y) \cdot \varphi^{-1}(b(x,y)).$$

From here $c(x,y) \cdot \varphi^{-1}(b(x,y)) = 1$, i.e. $c(x,y), b(x,y) \in k^*$. Therefore

$$\varphi(f(x)) = \lambda f(x),\ \lambda \in k^*.$$

Thus $f(\varphi(x)) = \lambda f(x)$. From here we get $deg(\varphi(x)) = 1$, i.e.

$$\varphi(x) = \alpha x + \beta,\ \alpha \in k^*,\ \beta \in k.$$

Let

$$\varphi(y) = \sum_{i \geq 0} y^i g_i(x) = g_0(x) + \sum_{i \geq 1} y^i g_i(x).$$

Using the ratio $\{\varphi(x), \varphi(y)\} = \varphi(f(x)) = f(\varphi(x))$ we get

$$\left\{\alpha x + \beta, g_0(x) + \sum_{i \geq 1} y^i g_i(x)\right\} = \sum_{i \geq 1} \alpha \left\{x, y^i\right\} g_i(x) = f(\alpha x + \beta).$$

We have

$$\left\{x, y^i\right\} = i f(x) y^{i-1}.$$

Therefore

$$\sum_{i \geq 1} i \alpha f(x) y^{i-1} g_i(x) = f(\alpha x + \beta),$$

i.e. $i\alpha f(x) g_i(x) = 0$ at $i > 1$ and $\alpha f(x) g_1(x) = f(\alpha x + \beta)$.

From here we get $g_i(x) = 0$ at $i > 1$. Therefore

$$\varphi(y) = g_0(x) + \gamma y,$$

where $\gamma = g_1(x) \in k^*$. Thus we have

$$\varphi: \begin{array}{l} x \to \alpha x + \beta \\ y \to \gamma y + g(x), \quad g(x) \in k[x]. \end{array}$$

From here it is easy to derive that φ is represented in the form $\varphi = \sigma_\alpha \tau$, where $\sigma_\alpha \in H$ and $\tau \in T$.

Now for any $\sigma_\alpha \in G_f$ and $\tau \in T$ we have

$$\sigma_\alpha \tau \sigma_\alpha^{-1}(x) = \sigma_\alpha \tau \left(\frac{x - \beta}{\alpha}\right) = \sigma_\alpha \left(\frac{x - \beta}{\alpha}\right) = x \in T;$$

$$\sigma_\alpha \tau \sigma_\alpha^{-1}(y) = \sigma_\alpha \tau \left(\frac{y}{\gamma}\right) = \sigma_\alpha \left(\frac{y}{\gamma} + h(x)\right) = y + h'(x) \in T;$$

i.e. T – a normal subgroup of the group G_f.

Obviously $H \cap T = id$. Thus G_f is a semi-direct product of the groups H and T. □

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$k[x, y]$ көпмүшеліктер алгебрасындағы Пуассон алгебраларының автоморфизмдер тобы

Пуассон алгебралары Гамильтон механикасында, симплектикалық геометрияда, сонымен қатар кванттық топтарды зерттеуде маңызды рөл атқарады. Қазіргі уақытта Пуассон алгебраларын Ресей, Франция, АҚШ, Бразилия, Аргентина, Болгария және тағы басқа елдердің көптеген математиктері зерттеуде. Мақаланың мақсаты қосымша құрылымды, яғни Пуассон жақшасы берілген $k[x, y]$ көпмүшеліктер алгебрасынң автоморфизмдері тобын, зерттеу болып табылады. Кез келген $f \in k\,[x, y]$ үшін $k\,[x, y]$ ассоциативті-коммутативті алгебрасын, $\{x, y\} = f$ ережесі арқылы Пуассон жақшасын анықтай отырып, P_f Пуассон алгебрасына айналдыруға болады. P_f Пуассон алгебрасының G_f автоморфизмдері тобының құрылымын f-ке тәуелділігін зерттейміз. G_f тобының толық сипаттамасы рангі 1-ден кіші немесе тең f көпмүшелігі үшін келтірілген. Бұл жұмыста барлық алгебралар сипаттаушысы 0 болатын кез келген k өрісінде қарастырылды.

Кілт сөздер: Пуассон алгебралары, көпмүшеліктер алгебралары, автоморфизмдер, қосымша құрылым.

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Группа автоморфизмов алгебры Пуассона на $k[x, y]$

Алгебры Пуассона играют ключевую роль в гамильтоновой механике, симплектической геометрии и также являются центральными в изучении квантовых групп. В настоящее время алгебры Пуассона исследуются многими математиками России, Франции, США, Бразилии, Аргентины, Болгарии и т.д. Целью настоящей работы является исследование группы автоморфизмов алгебры многочленов $k[x, y]$ с дополнительной структурой, со скобкой Пуассона. Для любого $f \in k\,[x, y]$ ассоциативно-коммутативную алгебру $k\,[x, y]$ можно превратить в алгебру Пуассона P_f, определяя скобку Пуассона правилом $\{x, y\} = f$. Мы изучили строение группы автоморфизмов G_f алгебры Пуассона P_f в зависимости от f. Полное описание группы G_f приведено для многочлена f ранга меньше или равно 1. Все алгебры в данной работе рассмотрены над произвольным полем k характеристики 0.

Ключевые слова: алгебры Пуассона, алгебры многочленов, автоморфизмы, дополнительная структура.

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