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Fourier–Price Coefficients of Class GM and Best Approximations of Functions in the Lorentz Space $L_{p\theta}[0,1)$, $1<p<+\infty$, $1<\theta<+\infty$

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Abstract—For polynomials in the Price system, we establish an inequality of different metrics in the Lorentz spaces. Applying this inequality, we prove a Hardy–Littlewood theorem for the Fourier–Price series with GM sequences of coefficients in the two-parameter Lorentz spaces and in the Nikol'skii–Besov spaces with a Price basis. We also study the behavior of the best approximations of functions by Price polynomials in the metric of the Lorentz space.

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1. INTRODUCTION

The Hardy–Littlewood theorem for trigonometric series with monotone Fourier coefficients in the Lebesgue spaces $L_p[0,2\pi)$, $1<p<+\infty$ (see [31]), has played an essential role in function theory. The sharpness of various statements has been proved and extreme functions of function classes have been constructed on the basis of this theorem.

The theorem has been developed in several directions. First, in the space $L_p[0,2\pi)$, $1<p<+\infty$, the condition of monotonicity of the sequence of trigonometric coefficients has been relaxed in various ways; namely, in [12], the sufficient condition of the Hardy–Littlewood theorem was established for trigonometric series with quasimonotone coefficients; in [15], the sufficient condition was established for trigonometric Fourier series with coefficients in the RBVS classes. In 2007, Tikhonov [27, Theorem 4.2] reproved the Hardy–Littlewood theorem with a new, weaker, monotonicity condition imposed on the coefficients of trigonometric series. Tikhonov showed that his class of sequences contains both monotone and quasimonotone sequences as well as RBVS sequences of coefficients. Moreover, it should be noted that D'yachenko in [5] also proved the Hardy–Littlewood theorem for a wide monotonicity class of Fourier coefficients.

Second, in [16], Nursultanov extended the Hardy–Littlewood theorem to multiple trigonometric series under the condition of generalized monotonicity defined by him; however, the assertion of the Hardy–Littlewood theorem remained valid only for $2\le p<+\infty$. For $1<p<2$, D'yachenko constructed an example of a function f_0 such that all hypotheses of Nursultanov's theorem for the sequence $\{\widehat{f_0}(k_1,\dots,k_n)\}$ are satisfied but $f_0\in L_p[0,2\pi)^n$.

Third, the Hardy–Littlewood theorem has been extended to more general spaces than the space $L_p[0,2\pi)$, $1<p<+\infty$. For example, in [20], Sagher established the Hardy–Littlewood theorem for trigonometric series in the two-parameter Lorentz space under the ordinary monotonicity condition. In [14, Ch. II, Theorem 9.3], the Hardy–Littlewood theorem was proved for a cosine series in symmetric spaces under the condition that the sequence of coefficients is monotone.

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Here we should also mention the papers [21, 18, 19, 9, 10, 6].

Fourth, the Hardy–Littlewood theorem has been established for series in other orthonormal systems different from the trigonometric ones, namely, for Walsh and Haar series as well as for series in multiplicative systems. For example, in [28] Timan and Tukhliev proved the Hardy–Littlewood theorem for the Price series in the space $L_q[0,1)$, $1 < q < +\infty$, under the condition that the coefficients are monotone and the generating sequence is bounded. G. Akishev proved the Hardy–Littlewood theorem in $L_p[0,1)$, $1 < p < +\infty$, for the Price series with quasimonotone coefficients without requiring the boundedness of the generating number sequence of the Price system (see [3, Theorem 10.1]). In [22], this theorem was established in the Lorentz space without requiring the boundedness of the generating sequence of the Price system.

Here we also mention another paper by Nursultanov [17], where he established the Hardy–Littlewood theorem for multiple Fourier series in regular systems introduced by him for $1 < p < 2$ under the condition of generalized monotonicity of the function and for $2 < p < +\infty$ under the condition of generalized monotonicity of the table of Fourier coefficients.

In the present paper, we obtain the Hardy–Littlewood theorem for Fourier–Price series with GM sequences (see [27]) of coefficients in the two-parameter Lorentz space and analyze the behavior of the best approximations of functions in the metric of the Lorentz space by means of Price polynomials.

In [13], Konyushkov presented two-sided estimates for the best trigonometric approximations of functions in the metric of the space $L_p[0,2\pi)$, $1 < p < +\infty$, in terms of Fourier coefficients under the condition of their monotonicity. In 1982, in the above-mentioned paper [12], Kokilashvili published similar estimates for the best trigonometric approximations in the metric of the space $L_p[0,2\pi)$, $1 < p < +\infty$, in terms of quasimonotone trigonometric Fourier coefficients. Leindler [15] established an upper estimate for the best trigonometric approximations in the metric of the space $L_p[0,2\pi)$, $1 < p < +\infty$, for RBVS sequences of coefficients. In [26], the best trigonometric approximations in the metric of the Lorentz space were estimated both from below and from above in terms of trigonometric Fourier coefficients under the condition of quasimonotonicity of the latter.

In [1], Agafonova presented an upper estimate for the best approximation of functions in the metric of the space $L_p[0,1)$, $1 < p < +\infty$, by Price polynomials in terms of Fourier–Price coefficients under the conditions that the generating sequence of the Price system is bounded and the Fourier–Price coefficients belong to the class RBVS or A_τ, $\tau \in \mathbb{R}$. Here $\{a_k\}_{k=1}^{+\infty} \in A_\tau$ for $\tau > 0$ means that $\{a_k k^{-\tau}\}_{k=1}^{+\infty}$ decreases and $\lim_{k\to+\infty} a_k = 0$, while for $\tau < 0$ it means that $\{a_k k^{|\tau|}\}_{k=1}^{+\infty}$ increases and $\lim_{k\to+\infty} a_k = 0$.

In [23], two-sided estimates for the best approximations in the metric of the Lorentz space by Price polynomials are obtained without constraints on the generating sequence of the Price system but under the condition that the sequence of coefficients of the Price series is quasimonotone.

In this paper, we present similar results but under a weaker condition imposed on the sequence of coefficients, namely, under the condition that the sequence of coefficients belongs to the class GM defined in [27].

2. DEFINITIONS AND AUXILIARY STATEMENTS

Let $\bar{p} = \{p_1, p_2, \dots\}$ be an arbitrary sequence of positive integers with $p_k \ge 2$, $k \ge 1$. The sequence $\{p_k\}$ defines a set G of integer sequences $\widetilde{x} = \{x_1, x_2, \dots, x_k, \dots\}$ with $0 \le x_k \le p_k - 1$. Set $m_0 = 1$ and $m_n = \prod_{k=1}^{n} p_k$. To each element $\widetilde{x} = \{x_k\}_{k=1}^{+\infty} \in G$, we assign a real number

$$x = \lambda(\widetilde{x}) = \sum_{k=1}^{+\infty} \frac{x_k}{m_k}, \qquad 0 \le x_k \le p_k - 1, \tag{2.1}$$

which lies in the interval $[0,1]$.

The set G is a group [3, 7].

Any positive integer n can be uniquely represented as

$$n = \sum_{k=0}^{l} n_k m_k,$$

where n_k are integers such that $0 \le n_k \le p_{k+1} - 1$, $k \in \mathbb{Z}^+$.

Equality (2.1) defines a mapping from the group G to the interval. This mapping is one-to-one at all points of the interval $[0, 1]$ except points of the form l/m_k, $0 \le l \le m_k - 1$, $k \in \mathbb{N}$. We will refer to points of the form l/m_k as *p_k-rational points* and to the other points as *p_k-irrational points of the interval* $[0, 1]$. To points of the form l/m_k, there correspond two types of expansions (2.1). One of them is finite, with $x_k = 0$ for all $k > n$, and the other is infinite, with $x_k = p_k - 1$ for all $k > n$.

If x is not p_k-rational, then the expansion (2.1) is a unique representation of x. If p_k-rational points are counted twice, then the mapping of the group G to the interval $[0, 1]$ by means of the representation (2.1) is one-to-one.

The interval $[0, 1)$ with p_k-rational points counted twice is called the *modified* interval $[0, 1)$; everywhere below, $[0, 1)$ stands for the modified interval $[0, 1)$.

Now, we define a multiplicative Price system $\Phi = \{\varphi_k(x)\}_{k=0}^{+\infty}$ on the interval $[0, 1)$ (see [3, 7]).

Set $\varphi_0(x) \equiv 1$, $x \in [0, 1)$, and

$$\varphi_{m_k}(x) = \exp\left\{\frac{2\pi i x_{k+1}}{p_{k+1}}\right\}, \qquad k \in \mathbb{Z}^+,$$

where x_k are the numbers from the expansion (2.1) of the point x. For integers $n \ge 1$, set

$$\varphi_n(x) = \prod_{k=0}^{r} \big[\varphi_{m_k}(x)\big]^{n_k}, \qquad x \in [0, 1).$$

The multiplicative Price system thus defined is orthonormal (see [7, 3]).

The *Fourier–Price series* of a function $f \in L_1[0, 1)$ is the Price series $\sum_{\nu=0}^{+\infty} a_\nu \varphi_\nu(x)$ whose coefficients are defined by the equality

$$a_\nu = \int_0^1 f(t)\varphi_\nu(t)\,dt, \qquad \nu \in \mathbb{Z}^+.$$

Definition 2.1 [25]**.** Let $f(x)$ be a nonnegative function defined on $[0, 1)$ that is measurable and finite almost everywhere on $[0, 1)$. Set

$$L_t = \{x \in [0, 1)\colon f(x) > t\}.$$

The function $\mu_f(t)$ whose value at t is equal to the Lebesgue measure $\mu(L_t)$ of the set L_t is called the *distribution of the function* $f(x)$. The function $\mu_f(t)$ is right continuous and does not increase on $(0, +\infty)$.

Definition 2.2 [25]**.** Let $f(x)$ be defined and Lebesgue measurable on $[0, 1)$. The function

$$f^*(\tau) = \inf\big\{t \in (0, +\infty)\colon\ \mu_{|f|}(t) \le \tau\big\}, \qquad \tau \in [0, +\infty),$$

is called the *nonincreasing rearrangement of the function* $|f(x)|$.

Definition 2.3. [25] Let $f(x)$ be a Lebesgue measurable function on $[0,1)$, $1 \le p < +\infty$, and $1 \le \theta < +\infty$. We say that $f(x)$ *belongs to the Lorentz space* $L_{p\theta}[0,1)$ if

$$\|f\|_{p\theta} = \left\{ \int\limits_0^1 t^{\frac{\theta}{p}-1} (f^*(t))^\theta \, dt \right\}^{\frac{1}{\theta}} < +\infty.$$

For $\theta = p$, by definition, $L_{pp}[0,1) = L_p[0,1)$ and $\|f\|_{pp} := \|f\|_p$.

For short, a linear aggregate of the Price system

$$T_n(x) = \sum_{k=0}^{n-1} a_k \varphi_k(x)$$

will be called a *Price polynomial of degree* n. Let $f \in L_{p\theta}[0,1)$, $1 \le p < +\infty$, and $1 \le \theta < +\infty$. The quantity

$$E_n(f)_{p\theta} = \inf\bigl\{\|f - T_l\|_{p\theta} \colon \{T_l(x)\},\ l \le n\bigr\}$$

is called the *best approximation* of a function f in the metric of the Lorentz space $L_{p\theta}[0,1)$ by Price polynomials of degree at most n.

Denote by M the class of nonnegative number sequences that decrease monotonically to zero. By QM (see [27]), denote the class of quasimonotone number sequences, i.e.,

$$\mathrm{QM} = \Bigl\{ a_n \in \mathbb{R} \colon \lim_{n\to+\infty} a_n = 0 \text{ and } \exists\, \tau \ge 0 \colon \frac{a_n}{n^\tau} \downarrow 0 \Bigr\}.$$

Definition 2.4 [27]. We say that a sequence of complex numbers $a = \{a_k\}_{k=1}^{+\infty}$ *belongs to the set* GM if the inequality

$$\sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \le C|a_n| \qquad \forall\, n \in \mathbb{N}$$

holds with a constant $C > 0$ independent of n.

According to [27], the set GM contains monotone and quasimonotone number sequences.

Below we will need the following property of number sequences that belong to GM:

$$a \in \mathrm{GM} \qquad \Rightarrow \qquad |a_k| \le C|a_n| \quad \forall\, k, n \in \mathbb{N},\ n \le k \le 2n. \tag{2.2}$$

Here $C > 0$ is independent of both n and k.

Lemma 2.1 [23]. *Let* $1 < p < q < +\infty$, $1 < \tau < +\infty$, *and* $1 < \theta < +\infty$, *and let* $\Phi_{2^l}(x) = \sum_{\nu=0}^{2^l-1} a_\nu \varphi_\nu(x)$ *be Price polynomials. Then, for* $l > k$, *one has*

$$\|\Phi_{2^l} - \Phi_{2^k}\|_{q\tau} \le c_{pq\theta\tau} \left\{ \sum_{s=k}^{l-1} 2^{s\tau(\frac{1}{p}-\frac{1}{q})} \|\Phi_{2^{s+1}} - \Phi_{2^s}\|_{p\theta}^\tau \right\}^{\frac{1}{\tau}}.$$

Here the coefficient $c_{pq\theta\tau} > 0$ *depends only on the indicated parameters.*

Theorem 2.1 [22]. *Let* $1 < p < +\infty$, $1 < \theta < +\infty$, *and* $a = \{a_\nu\}_{\nu=0}^{+\infty} \in \mathrm{QM}$. *The numbers* a_ν *are the Fourier–Price coefficients of some function* $f \in L_{p\theta}[0,1)$ *if and only if the series*

$$\sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta$$

converges. In this case, the following inequality holds:

$$c'_{p\theta}\left\{a_0^\theta+\sum_{\nu=1}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}a_\nu^\theta\right\}^{\frac{1}{\theta}}\le\|f\|_{p\theta}\le c_{p\theta}\left\{a_0^\theta+\sum_{\nu=1}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}a_\nu^\theta\right\}^{\frac{1}{\theta}}.$$

Here the coefficients $c_{p\theta}, c'_{p\theta}>0$ depend only on the indicated parameters.

Definition 2.5 [16]**.** A sequence of positive numbers $\{a_k\}_{k\in\mathbb{Z}^n}$ is said to be *generalized monotone in* $\mathbb{Z}^n$ if for any $m\in\mathbb{Z}^n$ we have

$$|a_m|\le C\frac{1}{|Q_m|}\left|\sum_{k\in Q_m}a_k\right|,\qquad Q_m=\left\{k\in\mathbb{Z}^n\colon\ 0\le k_j\operatorname{sgn}m_j\le|m|\right\}.$$

The condition of generalized monotonicity is weaker than monotonicity and quasimonotonicity.

Theorem 2.2 [16, Corollary 5]**.** *Let $2\le p<+\infty$ and $f\sim\sum_{k\in\mathbb{Z}^n}\widehat{f}(k)e^{ikx}$, where $\{\widehat{f}(k)\}_{k\in\mathbb{Z}^n}$ is generalized monotone in $\mathbb{Z}^n$. Then $f\in L_p$ if and only if*

$$\sum_{k_1=1}^{+\infty}\dots\sum_{k_n=1}^{+\infty}k_1^{p-2}\dots k_n^{p-2}\left|\widehat{f}(k_1,\dots,k_n)\right|^p<+\infty.$$

Lemma 2.2 [25]**.** *Let $f(x)$ and $g(x)$ be nonnegative functions defined in $\Omega\subset\mathbb{R}$. If $f\cdot g\in L_1(\Omega)$, then the following inequality holds*:

$$\int_\Omega f(x)g(x)\,dx\le\int_0^{\mu(\Omega)}f^*(t)g^*(t)\,dt.$$

Lemma 2.3. *Let $1<p<+\infty$, $1<\theta<+\infty$, and $S_n(f;x)$ be the partial Fourier–Price sum of a function $f\in L_{p\theta}[0,1)$. Then*

$$\|S_n(f)\|_{p\theta}\le c_{p\theta}\|f\|_{p\theta}\qquad\forall f\in L_{p\theta}[0,1).$$

Here the coefficient $c_{p\theta}>0$ depends only on the indicated parameters.

Proof. In [30], Yong showed that for $1<p<+\infty$ the partial sums of the Fourier–Price series satisfy the inequality

$$\|S_n(f)\|_p\le c_p\|f\|_p\qquad\forall f\in L_p[0,1),\quad\forall\, n\in\mathbb{N}.$$

Using this assertion and the Marcinkiewicz interpolation theorem [2], we can verify that the operator of partial Fourier–Price sum is bounded in the Lorentz spaces $L_{p\theta}[0,1)$, $1<p<+\infty$, $1<\theta<+\infty$, as well. □

Lemma 2.4 [4]**.** *Let k and l be positive integers, $k>l$, and $D_{k,l}(x)=\sum_{\nu=l}^{k-1}\varphi_\nu(x)$ be the difference of Dirichlet kernels. Then the inequality*

$$\|D_{k,l}\|_p\le c_p(k-l)^{1-\frac{1}{p}}$$

holds for $1<p<+\infty$ with a constant c_p depending only on p.

Theorem 2.3. *Let numbers p and θ be such that $1<p<+\infty$ and $1\le\theta\le+\infty$. Then the following inequality holds for linear aggregates of multiplicative Price systems*:

$$\max_{t\in[0,1)}|T_n(t)|\le c_p n^{\frac{1}{p}}\|T_n\|_{p\theta}. \tag{2.3}$$

Proof. For a function $f \in L_p[0,1)$, $1 < p < +\infty$, we have the integral representation

$$S_n(f;x) = \int\limits_0^1 f(x) D_n(y \dot{-} x)\, dy,$$

where $D_n(x)$ is the Dirichlet kernel of the Fourier–Price series. Here $\dot{-}$ denotes the group operation defined by the Price system. Applying Hölder's inequality for $\frac{1}{p} + \frac{1}{p'} = 1$ and Lemma 2.4, we obtain

$$|S_n(f;x)| \le \int\limits_0^1 |f(x)| \cdot |D_n(y \dot{-} x)|\, dy \le \left\{ \int\limits_0^1 |f(x)|^p\, dy \right\}^{\frac{1}{p}} \left\{ \int\limits_0^1 |D_n(y \dot{-} x)|^{p'}\, dy \right\}^{\frac{1}{p'}}$$

$$= \|f\|_p \|D_n\|_{p'} \le c_p n^{\frac{1}{p}} \|f\|_p \qquad \forall x \in [0,1).$$

Here the constant $c_p > 0$ depends only on p.

Let $p_0 \neq p_1$, $1 \le p_i < +\infty$, $i = 0,1$, and

$$\|S_n(f)\|_\infty \le c_{p_0} n^{\frac{1}{p_0}} \|f\|_{p_0} \qquad \forall f \in L_{p_0}[0,1),$$

$$\|S_n(f)\|_\infty \le c_{p_1} n^{\frac{1}{p_1}} \|f\|_{p_1} \qquad \forall f \in L_{p_1}[0,1),$$

i.e.,

$$\|S_n\|_{p_0 \to \infty} \le c_{p_0} n^{\frac{1}{p_0}} = M_0 \qquad \text{and} \qquad \|S_n\|_{p_1 \to \infty} \le c_{p_1} n^{\frac{1}{p_1}} = M_1.$$

Then, by the Marcinkiewicz interpolation theorem (see [2, Theorem 5.3.2]), we have

$$\|S_n(f)\|_{p\theta \to \infty} \le M_0^{1-\alpha} M_1^{\alpha} = c_{p_0}^{1-\alpha} c_{p_1}^{\alpha} \big(n^{\frac{1}{p_0}}\big)^{1-\alpha} \big(n^{\frac{1}{p_1}}\big)^{\alpha} = B_{p_0 p_1 \alpha} n^{\frac{1}{p}},$$

where $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$, $0 < \alpha < 1$, and $1 \le \theta \le +\infty$.

Thus, the inequality

$$\|S_n(f)\|_\infty \le B_{p_0 p_1 \alpha} n^{\frac{1}{p}} \|f\|_{p\theta}$$

holds for all $f \in L_{p\theta}[0,1)$, $1 < p < +\infty$, $1 \le \theta \le +\infty$.

Setting $f(x) = T_n(x)$, in view of the orthonormality of the Price system, we obtain

$$\|T_n\|_\infty = \|S_n(T_n)\|_\infty \le B_{p\alpha} n^{\frac{1}{p}} \|T_n\|_{p\theta}.$$

Hence,

$$\max_{x \in [0,1)} |T_n(x)| \le c_p n^{\frac{1}{p}} \|T_n\|_{p\theta}, \qquad 1 < p < +\infty, \quad 1 \le \theta \le +\infty. \qquad \square$$

Theorem 2.4. *Let numbers p, q, θ, and τ be such that $1 < p < q \le +\infty$, $1 \le \theta$, and $\tau \le +\infty$. Then the following inequality holds for linear aggregates $T_n(x) = \sum_{k=0}^{n-1} a_k \varphi_k(x)$ of the multiplicative Price system:*

$$\|T_n\|_{q\tau} \le c_{pq\tau} n^{\frac{1}{p} - \frac{1}{q}} \|T_n\|_{p\theta},$$

where the constant $c_{pq\tau} > 0$ depends only on the indicated parameters.

Proof. For $q = +\infty$ and $\tau = +\infty$, the claim follows from Theorem 2.3. Now, let $1 < q < +\infty$ and $1 \le \tau < +\infty$. Then

$$\|T_n\|_{q\tau}^{\tau} = \frac{\tau}{q} \int\limits_0^1 t^{\frac{\tau}{q} - 1} (T_n^*(t))^\tau\, dt = \frac{\tau}{q} \int\limits_0^{1/n} t^{\frac{\tau}{q} - 1} (T_n^*(t))^\tau\, dt + \frac{\tau}{q} \int\limits_{1/n}^1 t^{\frac{\tau}{q} - 1} (T_n^*(t))^\tau\, dt = I_1 + I_2.$$

The integrals I_1 and I_2 will be considered separately. Applying Theorem 2.3, we obtain

$$I_1 \le \left(\max_{x\in[0,1)}|T_n(x)|\right)^\tau \frac{\tau}{q}\int\limits_0^{1/n} t^{\frac{\tau}{q}-1}\,dt \le n^{\tau(\frac{1}{p}-\frac{1}{q})}\|T_n\|_{p\theta}^\tau.$$

Since $T_n^*(t)$ does not increase on $[0,1)$, for $1<p,\theta<+\infty$ and $t\in[0,1)$ we have

$$t^{\frac{1}{p}}T_n^*(t) = T_n^*(t)\left\{\frac{\theta}{p}\int\limits_0^t u^{\frac{\theta}{p}-1}\,du\right\}^{\frac{1}{\theta}} \le \left\{\frac{\theta}{p}\int\limits_0^t u^{\frac{\theta}{p}-1}(T_n^*(u))^\theta\,du\right\}^{\frac{1}{\theta}} \le \left\{\frac{\theta}{p}\int\limits_0^1 u^{\frac{\theta}{p}-1}(T_n^*(u))^\theta\,du\right\}^{\frac{1}{\theta}}$$
$$= \|T_n\|_{p\theta}.$$

Hence, for $1<p<+\infty$ and $1\le\theta\le+\infty$,

$$\sup_{t\in[0,1)} t^{\frac{1}{p}}T_n^*(t) \le \|T_n\|_{p\theta}.$$

Under the conditions $1<p<q<+\infty$, $1\le\theta\le+\infty$, and $1\le\tau<+\infty$, we obtain

$$I_2 = \frac{\tau}{q}\int\limits_{1/n}^1 t^{\frac{\tau}{q}-\frac{\tau}{p}-1}\big(t^{\frac{1}{p}}T_n^*(t)\big)^\tau\,dt \le \left(\sup_{t\in[0,1)} t^{\frac{1}{p}}T_n^*(t)\right)^\tau \frac{\tau}{q}\int\limits_{1/n}^1 t^{\frac{\tau}{q}-\frac{\tau}{p}-1}\,dt \le c_{pq\tau}\,n^{\tau(\frac{1}{p}-\frac{1}{q})}\|T_n\|_{p\theta}^\tau.$$

Then

$$\|T_n\|_{q\tau}^\tau \le n^{\tau(\frac{1}{p}-\frac{1}{q})}\|T_n\|_{p\theta}^\tau + c_{pq\tau}n^{\tau(\frac{1}{p}-\frac{1}{q})}\|T_n\|_{p\theta}^\tau \le (1+c_{pq\tau})n^{\tau(\frac{1}{p}-\frac{1}{q})}\|T_n\|_{p\theta}^\tau,$$

where $1<p<q<+\infty$, $1\le\theta\le+\infty$, and $1\le\tau<+\infty$.

Now, let $\tau=+\infty$ and $1<q<+\infty$. Then, for $t\in[0,1)$ we have

$$t^{\frac{1}{q}}T_n^*(t) = T_n^*(t)\left\{\frac{\theta}{q}\int\limits_0^t u^{\frac{\theta}{q}-1}\,du\right\}^{\frac{1}{\theta}} \le \left\{\frac{\theta}{q}\int\limits_0^t u^{\frac{\theta}{q}-1}(T_n^*(u))^\theta\,du\right\}^{\frac{1}{\theta}} \le \left\{\frac{\theta}{q}\int\limits_0^1 u^{\frac{\theta}{q}-1}(T_n^*(u))^\theta\,du\right\}^{\frac{1}{\theta}}$$
$$\le \left\{\frac{\theta}{q}\int\limits_0^{1/n} u^{\frac{\theta}{q}-1}(T_n^*(u))^\theta\,du\right\}^{\frac{1}{\theta}} + \left\{\frac{\theta}{q}\int\limits_{1/n}^1 u^{\frac{\theta}{q}-1}(T_n^*(u))^\theta\,du\right\}^{\frac{1}{\theta}} = U_1+U_2.$$

By Theorem 2.3,

$$U_1 \le \max_{x\in[0,1)}|T_n(x)|\left\{\frac{\theta}{q}\int\limits_0^{1/n} u^{\frac{\theta}{q}-1}\,du\right\}^{\frac{1}{\theta}} \le n^{\frac{1}{p}-\frac{1}{q}}\|T_n\|_{p\theta}.$$

For the same reasons,

$$U_2 \le \sup_{t\in[0,1)} t^{\frac{1}{p}}T_n^*(t)\left\{\frac{\theta}{q}\int\limits_{1/n}^1 u^{\frac{\theta}{q}-\frac{\theta}{p}-1}\,du\right\}^{\frac{1}{\theta}} \le c_{pq\theta}\,n^{\frac{1}{p}-\frac{1}{q}}\|T_n\|_{p\theta}.$$

Thus,

$$\|T_n\|_{q,+\infty} = \sup_{t\in[0,1)} t^{\frac{1}{q}}T_n^*(t) \le U_1+U_2 \le c'_{pq\theta}\,n^{\frac{1}{p}-\frac{1}{q}}\|T_n\|_{p\theta}. \quad \square$$

Lemma 2.5. *Let $D_n(x)$ be the Dirichlet kernel of the Price system and let $1 < p < +\infty$ and $1 \le \theta \le +\infty$. Then*

$$\|D_n\|_{p\theta} \le c_{p\theta} n^{1-\frac{1}{p}}$$

for all $n \in \mathbb{N}$, where the constant $c_{p\theta} > 0$ depends only on the indicated parameters.

Proof. Let $2 < p < +\infty$ and $1 \le \theta \le +\infty$. Then, by Theorem 2.4 and Parseval's identity, we have

$$\left\|\sum_{k=0}^{n-1} \varphi_k\right\|_{p\theta} \le c_{p\theta} n^{\frac{1}{2}-\frac{1}{p}} \left\|\sum_{k=0}^{n-1} \varphi_k\right\|_{2} \le c_{p\theta} n^{\frac{1}{2}-\frac{1}{p}} n^{\frac{1}{2}} = c_{p\theta} n^{1-\frac{1}{p}}.$$

If $1 < r < p \le 2$ and $1 \le \theta \le +\infty$, then, in exactly the same way, applying Theorem 2.4 and Lemma 2.4, we establish

$$\left\|\sum_{k=0}^{n-1} \varphi_k\right\|_{p\theta} \le c_{p\theta r} n^{\frac{1}{r}-\frac{1}{p}} \left\|\sum_{k=0}^{n-1} \varphi_k\right\|_{r} \le c_{p\theta r} n^{\frac{1}{r}-\frac{1}{p}} n^{1-\frac{1}{r}} = c_{p\theta r} n^{1-\frac{1}{p}}. \qquad \square$$

Lemma 2.6. *Let $1 < p < +\infty$, $1 < \theta < +\infty$, $f \in L_{p\theta}[0,1)$, and $\sum_{k=0}^{+\infty} a_k\varphi_k(x)$ be its Fourier–Price series. Then*

$$\left|\sum_{k=n}^{2n-1} a_k\right| \le c_{p\theta} n^{\frac{1}{p}} E_n(f)_{p\theta}$$

for all $n \in \mathbb{N}$. Here the coefficient $c_{p\theta} > 0$ depends only on the indicated parameters.

Proof. The proof is standard. Let $T_n(x) = \sum_{k=0}^{n-1} a_k\varphi_k(x)$ be the polynomial of best approximation of the function $f \in L_{p\theta}[0,1)$ with respect to the Price system. Since the Price system is orthonormal, we have

$$\sum_{k=n}^{2n-1} a_k = \sum_{k=n}^{2n-1} \int_0^1 f(t)\varphi_k(t)\,dt = \int_0^1 (f(t) - T_n(t)) \sum_{k=n}^{2n-1} \varphi_k(t)\,dt = \int_0^1 (f(t) - T_n(t))(D_{2n}(t) - D_n(t))\,dt,$$

where $D_n(x) = \sum_{k=0}^{n-1} \varphi_k(t)$ is the Dirichlet kernel of the Price system. Applying Hölder's inequality and Lemma 2.5, we then immediately obtain

$$\left|\sum_{k=n}^{2n-1} a_k\right| \le \|f - T_n\|_{p\theta}\big(\|D_{2n}\|_{p'\theta'} + \|D_n\|_{p'\theta'}\big) \le c_{p\theta} n^{\frac{1}{p}} E_n(f)_{p\theta}. \qquad \square$$

Lemma 2.7 [11]**.** *Let $\gamma < 1$, $\beta > 1$, and $b_k \ge 0$. Then*

$$\sum_{k=1}^{+\infty} k^{-\gamma} \left(\sum_{\nu=k}^{+\infty} b_\nu\right)^{\beta} \le c_{\gamma\beta} \sum_{k=1}^{+\infty} k^{-\gamma}(k b_k)^{\beta}.$$

Lemma 2.8 [8]**.** *Let $r > 0$ and a sequence $\{b_l\}_{l=0}^{+\infty}$ be such that $b_0 = 1$ and $b_{l+1}/b_l \ge \alpha > 1$. Then the following inequality is valid for any numbers $q > 0$ and $\{a_k\}_{k=0}^{+\infty}$, $a_k \ge 0$:*

$$\sum_{l=0}^{+\infty} b_l^r \left(\sum_{k=l}^{+\infty} a_k\right)^{q} \le c_{rq} \sum_{l=0}^{+\infty} b_l^r a_l^q.$$

Here the coefficient $c_{rq} > 0$ depends only on the indicated parameters.

Definition 2.6. Let $1 \le p \le +\infty$, $1 \le \theta \le +\infty$, and $r > 0$. A function $f \in B^r_{p\theta}(\varphi;[0,1))$ is said to *belong to the Nikol'skii–Besov space with the Price basis* if $f \in L_p[0,1)$ and the following quantity is finite:

$$\bigl|f; B^r_{p\theta}(\varphi;[0,1))\bigr| = \|f\|_p + \left\{\sum_{k=0}^{+\infty} 2^{k\theta r} E^\theta_{2^k}(f)_p\right\}^{\frac{1}{\theta}} \qquad \text{if} \quad 1 \le \theta < +\infty,$$

$$\bigl|f; B^r_{p\infty}(\varphi;[0,1))\bigr| = \|f\|_p + \sup\bigl\{2^{kr} E_{2^k}(f)_p \colon k \in \mathbb{Z}^+\bigr\} \qquad \text{if} \quad \theta = +\infty.$$

Here $E_{2^k}(f)_p$ is the best approximation of the function $f \in L_p[0,1)$ by the Price polynomials (see [24]).

3. BEST APPROXIMATIONS OF FUNCTIONS $f \in L_{p\theta}[0,1)$ WITH GM FOURIER COEFFICIENTS

Theorem 3.1. *Let $1 < p < +\infty$, $1 < \theta < +\infty$, and $\bar a = \{a_\nu\}_{\nu=0}^{+\infty} \in \mathrm{GM}$ be a positive sequence. Then $\bar a$ is the sequence of Fourier–Price coefficients of some function $f \in L_{p\theta}[0,1)$ if and only if the series*

$$\sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a^\theta_\nu$$

converges. In this case, the inequality

$$c'_{p\theta}\left\{a^\theta_0 + \sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a^\theta_\nu\right\}^{\frac{1}{\theta}} \le \|f\|_{p\theta} \le c_{p\theta}\left\{a^\theta_0 + \sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a^\theta_\nu\right\}^{\frac{1}{\theta}}$$

holds. Here the coefficients $c_{p\theta}, c'_{p\theta} > 0$ depend only on the indicated parameters.

Proof. *Sufficiency.* Using the sequence $\bar a$, we construct a Price series $\sum_{\nu=0}^{+\infty} a_\nu \varphi_\nu(x)$, and let $S_{2^k} = \sum_{\nu=0}^{2^k-1} a_\nu \varphi_\nu$ be its partial sum. Take a number q, $1 < q < p < +\infty$. Then, by Lemma 2.1, we have

$$\|S_{2^l} - S_{2^k}\|_{p\theta} \le c_{pq\theta}\left\{\sum_{n=k}^{l-1} 2^{n\theta(\frac{1}{q}-\frac{1}{p})}\left\|\sum_{\nu=2^n}^{2^{n+1}-1} a_\nu\varphi_\nu\right\|^\theta_{q\theta}\right\}^{\frac{1}{\theta}}. \tag{3.1}$$

Applying Abel's transformation and Lemma 2.5, we obtain

$$\left\|\sum_{\nu=2^n}^{2^{n+1}-1} a_\nu\varphi_\nu\right\|_{q\theta} \le \sum_{\nu=2^n}^{2^{n+1}-2} |a_\nu - a_{\nu+1}| \left\|\sum_{s=0}^{\nu}\varphi_s\right\|_{q\theta} + a_{2^{n+1}-1}\left\|\sum_{s=0}^{2^{n+1}-1}\varphi_s\right\|_{q\theta} + a_{2^n}\left\|\sum_{s=0}^{2^n-1}\varphi_s\right\|_{q\theta}$$

$$\le c_{q\theta}\left\{\sum_{\nu=2^n}^{2^{n+1}-1} |a_\nu - a_{\nu+1}| + a_{2^{n+1}-1} + a_{2^n}\right\}\cdot 2^{n(1-\frac{1}{q})}.$$

Since $a \in \mathrm{GM}$, we have $\sum_{\nu=2^n}^{2^{n+1}-1}|a_\nu - a_{\nu+1}| \le c a_{2^n}$ for all $n \in \mathbb{N}$ according to property (2.2) of the sequences in GM. Hence,

$$\left\|\sum_{\nu=2^n}^{2^{n+1}-1} a_\nu\varphi_\nu\right\|_{q\theta} \le D_{q\theta}\, a_{2^n}\cdot 2^{n(1-\frac{1}{q})} \qquad \forall\, n \in \mathbb{Z}^+.$$

Substituting the estimate obtained into (3.1), we arrive at

$$\|S_{2^l} - S_{2^k}\|_{p\theta} \le D'_{pq\theta}\left\{\sum_{n=k}^{l-1} 2^{n\theta(1-\frac{1}{p})} a^\theta_{2^n}\right\}^{\frac{1}{\theta}} \qquad \forall\, l,k \in \mathbb{N}\colon\ l > k.$$

Taking into account the property of the sequences $a \in \mathrm{GM}$, we have

$$\sum_{\nu=2^n}^{2^{n+1}-1} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta \geq (a_{2^n})^\theta \sum_{\nu=2^n}^{2^{n+1}-1} \nu^{\theta(1-\frac{1}{p})-1} \geq b_{p\theta}\, a_{2^n}^\theta \cdot 2^{n\theta(1-\frac{1}{p})} \qquad \forall\, n \in \mathbb{N}.$$

Then it follows from (3.1) that

$$\|S_{2^l} - S_{2^k}\|_{p\theta} \leq c_{p\theta} \left\{ \sum_{\nu=2^k}^{2^l-1} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta \right\}^{\frac{1}{\theta}}. \tag{3.2}$$

By the hypothesis of the theorem, the series $\sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta$ converges. Therefore,

$$\|S_{2^l} - S_{2^k}\|_{p\theta} \to 0 \qquad \text{as} \quad \min\{k,l\} \to +\infty.$$

As the space $L_{p\theta}[0,1)$ is complete, there exists a function $f \in L_{p\theta}[0,1)$ such that $\|f - S_{2^k}\|_{p\theta} \to 0$ as $k \to +\infty$. Let $\{b_\nu(f)\}_{\nu=0}^{+\infty}$ be the sequence of Fourier coefficients of the function $f \in L_{p\theta}[0,1)$, $1 < p < +\infty$, $1 < \theta < +\infty$. Applying Hölder's inequality, for $2^k > n$ we have

$$\begin{aligned}
|b_n(f) - a_n| &\leq \int_0^1 |f(x) - S_{2^k}(x)| \cdot |\varphi_n(x)|\, dx \leq \int_0^1 (f - S_{2^k})^*(t) \varphi_n^*(t)\, dt \\
&\leq \left\{ \int_0^1 t^{\frac{\theta}{p}-1} \big((f - S_{2^k})^*(t)\big)^\theta\, dt \right\}^{\frac{1}{\theta}} \left\{ \int_0^1 t^{\frac{\theta'}{p'}-1} (\varphi_n^*(t))^{\theta'}\, dt \right\}^{\frac{1}{\theta'}} \\
&\leq \|f - S_{2^k}\|_{p\theta} \left\{ \int_0^1 t^{\frac{\theta'}{p'}-1}\, dt \right\}^{\frac{1}{\theta'}} = c_{p\theta} \|f - S_{2^k}\|_{p\theta} \to 0 \qquad \text{as} \quad k \to +\infty.
\end{aligned}$$

This means that $b_n(f) = a_n$, $n \in \mathbb{Z}^+$. Passing to the limit as $l \to +\infty$ for $k = 0$ in inequality (3.2), we obtain

$$\left\| \sum_{\nu=1}^{+\infty} a_\nu \varphi_\nu \right\|_{p\theta} \leq c_{pq\theta} \left\{ \sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta \right\}^{\frac{1}{\theta}}.$$

Since $f(x) \sim a_0 + \sum_{\nu=1}^{+\infty} a_\nu \varphi_\nu(x)$, we have $f(x) - a_0 \sim \sum_{\nu=1}^{+\infty} a_\nu \varphi_\nu(x)$. In view of the properties of the norm,

$$\|f\|_{p\theta} - c_{p\theta}|a_0| \leq \|f - a_0\|_{p\theta} \leq \left\| \sum_{\nu=1}^{+\infty} a_\nu \varphi_\nu \right\|_{p\theta} \leq c_{pq\theta} \left\{ \sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta \right\}^{\frac{1}{\theta}}.$$

Hence, $\|f\|_{p\theta} \leq c_{p\theta} \big\{ a_0^\theta + \sum_{\nu=1}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta \big\}^{\frac{1}{\theta}}$.

Necessity. Let $f \in L_{p\theta}[0,1)$ and $S_m(f;x)$ be a partial sum of the Fourier–Price series. Then, applying Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned}
\int_0^1 S_m(f;x) g(x)\, dx &\leq \int_0^1 S_m^*(t) g^*(t)\, dt = \int_0^1 \Big(t^{\frac{1}{p}-\frac{1}{\theta}} S_m^*(t) \Big) \Big(t^{\frac{1}{p'}-\frac{1}{\theta'}} g^*(t) \Big)\, dt \\
&\leq \left\{ \int_0^1 t^{\frac{\theta}{p}-1} (S_m^*(t))^\theta\, dt \right\}^{\frac{1}{\theta}} \left\{ \int_0^1 t^{\frac{\theta'}{p'}-1} (g^*(t))^{\theta'}\, dt \right\}^{\frac{1}{\theta'}} = \|S_m\|_{p\theta} \|g\|_{p'\theta'}.
\end{aligned}$$

This implies that

$$\|S_m\|_{p\theta} \geq \sup\left\{\int\limits_0^1 S_m(f;x)g(x)\,dx\colon\ \|g\|_{p'\theta'} \leq 1\right\}. \tag{3.3}$$

Consider the function $g_m(x) = \sum_{k=0}^{+\infty} c_k(g_m)\varphi_k(x)$ with

$$c_0(g_m) = a_0^{\theta-1}\left(a_0 + \sum_{\nu=1}^{m} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta\right)^{-\frac{1}{\theta'}}$$

and

$$c_k(g_m) = \begin{cases} a_k^{\theta-1} k^{\theta(1-\frac{1}{p})-1}\left(a_0^\theta + \sum\limits_{\nu=1}^{m} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta\right)^{-\frac{1}{\theta'}}, & k = 1,\dots,m, \\ 0, & k = m+1, m+2, \dots. \end{cases}$$

Let us show that the sequence $\{c_k(g_m)\}$ also belongs to GM. By the hypothesis, $\{a_k\} \in \mathrm{GM}$. For the sake of brevity, we introduce the notation

$$B_m = \left(a_0^\theta + \sum_{k=1}^{m} k^{\theta(1-\frac{1}{p})-1} a_k^\theta\right)^{-\frac{1}{\theta'}}.$$

Then

$$\begin{aligned}
\sum_{\nu=n}^{2n-1} |c_\nu - c_{\nu+1}| &= B_m \sum_{\nu=n}^{2n-1} \left|a_\nu^{\theta-1}\nu^{\theta(1-\frac{1}{p})-1} - a_{\nu+1}^{\theta-1}(\nu+1)^{\theta(1-\frac{1}{p})-1}\right| \\
&\leq B_m\left\{\sum_{\nu=n}^{2n-1}\left|a_\nu^{\theta-1}\nu^{\theta(1-\frac{1}{p})-1} - a_{\nu+1}^{\theta-1}\nu^{\theta(1-\frac{1}{p})-1} + a_{\nu+1}^{\theta-1}\nu^{\theta(1-\frac{1}{p})-1} - a_{\nu+1}^{\theta-1}(\nu+1)^{\theta(1-\frac{1}{p})-1}\right|\right\} \\
&\leq B_m\left\{\sum_{\nu=n}^{2n-1} \nu^{\theta(1-\frac{1}{p})-1}\left|a_\nu^{\theta-1} - a_{\nu+1}^{\theta-1}\right| + \sum_{\nu=n}^{2n-1} a_{\nu+1}^{\theta-1}\left|\nu^{\theta(1-\frac{1}{p})-1} - (\nu+1)^{\theta(1-\frac{1}{p})-1}\right|\right\} \\
&\leq B_m\left\{c_{p\theta} n^{\theta(1-\frac{1}{p})-1} \sum_{\nu=n}^{2n-1}\left|a_\nu^{\theta-1} - a_{\nu+1}^{\theta-1}\right| + c'_{p\theta} a_n^{\theta-1} \sum_{\nu=n}^{2n-1} \nu^{\theta(1-\frac{1}{p})-2}\right\}.
\end{aligned}$$

By Lagrange's mean value theorem, we obtain

$$\left|a_\nu^{\theta-1} - a_{\nu+1}^{\theta-1}\right| = (\theta-1)|a_\nu - a_{\nu+1}|\xi_\nu^{\theta-2},$$

where ξ lies between the numbers a_ν and $a_{\nu+1}$. Since $a \in \mathrm{GM}$, the following estimate is valid:

$$\sum_{\nu=n}^{2n-1}\left|a_\nu^{\theta-1} - a_{\nu+1}^{\theta-1}\right| \leq (\theta-1)\sum_{\nu=n}^{2n-1}|a_\nu - a_{\nu+1}| a_n^{\theta-2} \leq c_\theta\, a_n^{\theta-1}.$$

Thus,

$$\sum_{\nu=n}^{2n-1}|c_\nu - c_{\nu+1}| \leq c_\theta\, B_m a_n^{\theta-1} n^{\theta(1-\frac{1}{p})-1} \qquad \forall\, n \in \mathbb{N}\colon\ 1 \leq n < m.$$

Now, let m be a positive integer in the interval $[n, 2n-1]$. Then, according to the definition of the sequence $\{c_k(g_m)\}$, we have

$$\sum_{\nu=n}^{2n-1}|c_\nu - c_{\nu+1}| = \sum_{\nu=n}^{m}|c_\nu - c_{\nu+1}| + \sum_{\nu=m+1}^{2n-1}|c_\nu - c_{\nu+1}| = \sum_{\nu=n}^{m-1}|c_\nu - c_{\nu+1}| + |c_m - c_{m+1}|$$
$$= \sum_{\nu=n}^{m-1}|c_\nu - c_{\nu+1}| + c_m.$$

It is clear that

$$c_m = B_m\, a_m^{\theta-1} m^{\theta(1-\frac{1}{p})-1} \le c^{\theta-1} B_m\, a_n^{\theta-1} c_{p\theta}\, n^{\theta(1-\frac{1}{p})-1} = c''_{p\theta}\, B_m\, a_n^{\theta-1} n^{\theta(1-\frac{1}{p})-1} = c''_{p\theta}\, c_n.$$

Next, taking into account the previous calculations, we get

$$\sum_{\nu=n}^{m-1}|c_\nu - c_{\nu+1}| \le B_m\Biggl\{\sum_{\nu=n}^{m-1}\nu^{\theta(1-\frac{1}{p})-1}\bigl|a_\nu^{\theta-1} - a_{\nu+1}^{\theta-1}\bigr| + \sum_{\nu=n}^{m-1}a_{\nu+1}^{\theta-1}\Bigl|\nu^{\theta(1-\frac{1}{p})-1} - (\nu+1)^{\theta(1-\frac{1}{p})-1}\Bigr|\Biggr\}$$
$$\le B_m\Biggl\{c'''_{p\theta}\, n^{\theta(1-\frac{1}{p})-1}\sum_{\nu=n}^{m-1}|a_\nu - a_{\nu+1}|a_n^{\theta-2} + c^{(\mathrm{iv})}_{p\theta}\, a_n^{\theta-1} n^{\theta(1-\frac{1}{p})-1}\Biggr\}$$
$$\le c^{(\mathrm{v})}_{p\theta}\, B_m\, a_n^{\theta-1} n^{\theta(1-\frac{1}{p})-1} = c^{(\mathrm{v})}_{p\theta}\, c_n$$

in the case of $m \in [n, 2n-1]$ as well. Thus, the sequence $\{c_k(g_m)\} = c(g_m)$ belongs to GM. Now, consider the series

$$c_0^{\theta'}(g_m) + \sum_{k=1}^{+\infty}k^{\theta'(1-\frac{1}{p'})-1}c_k^{\theta'}(g_m) = \Biggl\{c_0^{\theta'} + \sum_{k=1}^{m}k^{\theta'(1-\frac{1}{p'})-1}k^{(\theta(1-\frac{1}{p})-1)\theta'}a_k^{(\theta-1)\theta'}\Biggr\}B_m^{\theta'}$$
$$= \Biggl\{a_0^\theta + \sum_{k=1}^{m}k^{\theta(1-\frac{1}{p})-1}a_k^\theta\Biggr\}\Biggl\{a_0^\theta + \sum_{k=1}^{m}k^{\theta(1-\frac{1}{p})-1}a_k^\theta\Biggr\}^{-1} = 1.$$

Hence, according to the proved part of the theorem,

$$\|g_m\|_{p'\theta'} \le D_{p\theta}\Biggl\{c_0^{\theta'}(g_m) + \sum_{k=1}^{+\infty}k^{\theta'(1-\frac{1}{p'})-1}c_k^{\theta'}(g_m)\Biggr\}^{\frac{1}{\theta'}} = D_{p\theta} < +\infty.$$

Let $\psi_m(x) = D_{p\theta}^{-1}\, g_m(x)$, $x \in [0,1)$. Then $\|\psi_m\|_{p'\theta'} \le 1$.

Next, taking into account the orthonormality of the Price system and (3.3), we have

$$\|f\|_{p\theta} \ge A_{p\theta}\|S_m(f)\|_{p\theta} \ge A_{p\theta}\sup\Biggl\{\int_0^1 S_m(f;x)g(x)\,dx\colon\ \|g\|_{p'\theta'} \le 1\Biggr\}$$
$$\ge A_{p\theta}\int_0^1 S_m(f;x)\psi_m(x)\,dx = A_{p\theta}D_{p\theta}^{-1}\int_0^1 S_m(f;x)g_m(x)\,dx = A_{p\theta}D_{p\theta}^{-1}\sum_{k=0}^{m}a_\nu(f)c_\nu(g_m)$$
$$= \Biggl\{a_0a_0^{\theta-1} + \sum_{\nu=1}^{m}a_\nu a_\nu^{\theta-1}\nu^{\theta(1-\frac{1}{p})-1}\Biggr\}\Biggl\{a_0^\theta + \sum_{k=1}^{m}k^{\theta(1-\frac{1}{p})-1}a_k^\theta\Biggr\}^{-\frac{1}{\theta'}} = \Biggl\{a_0^\theta + \sum_{\nu=1}^{m}a_\nu^\theta\,\nu^{\theta(1-\frac{1}{p})-1}\Biggr\}^{\frac{1}{\theta}}$$

for $m \in \mathbb{N}$. Thus,

$$\left\{a_0^\theta + \sum_{\nu=1}^{+\infty} a_\nu^\theta \nu^{\theta(1-\frac{1}{p})-1}\right\}^{\frac{1}{\theta}} \le A_{p\theta}^{-1} D_{p\theta}\|f\|_{p\theta}. \quad \square$$

Remark 3.1. During the proof of the theorem, no constraint was imposed on the generating sequence of the Price system.

Corollary 3.1. *Let $1 < p < +\infty$ and a positive sequence $\bar{a} = \{a_\nu\}_{\nu=0}^{+\infty}$ belong to* GM. *Then $\bar{a}$ is the sequence of Fourier–Price coefficients of some function $f \in L_p[0,1)$ if and only if the series*

$$\sum_{\nu=1}^{+\infty} \nu^{p-2} a_\nu^p$$

converges. In this case, the following inequality holds:

$$c_p'\left\{a_0^p + \sum_{\nu=1}^{+\infty} \nu^{p-2} a_\nu^p\right\}^{\frac{1}{p}} \le \|f\|_p \le c_p\left\{a_0^p + \sum_{\nu=1}^{+\infty} \nu^{p-2} a_\nu^p\right\}^{\frac{1}{p}}.$$

Here the coefficients $c_p, c_p' > 0$ depend only on p.

Theorem 3.2. *Let $f \in L_{p\theta}[0,1)$, where $1 < p < +\infty$ and $1 < \theta < +\infty$, and let $f(x) \sim \sum_{k=0}^{+\infty} a_k\varphi_k$ be its Fourier–Price series such that the positive sequence of Fourier–Price coefficients $\{a_k\}$ belongs to* GM. *Then the following inequality holds*:

$$E_n(f)_{p\theta} \le c_{p\theta}\left\{n^{1-\frac{1}{p}} a_n + \left(\sum_{k=n}^{+\infty} a_k^\theta k^{\theta(1-\frac{1}{p})-1}\right)^{\frac{1}{\theta}}\right\},$$

where the coefficient $c_{p\theta} > 0$ depends only on the indicated parameters.

Proof. Let $f \in L_{p\theta}[0,1)$, $f(x) \sim \sum_{k=0}^{+\infty} a_k\varphi_k(x)$ be its Fourier–Price series, and $S_n(f;x) = \sum_{k=0}^{n-1} a_k\varphi_k(x)$ be a partial sum of this series. Let $D_n(x) = \sum_{k=0}^{n-1}\varphi_k(x)$ be the Dirichlet kernel of the Price system. Now, consider the auxiliary Price polynomial $q_n(x) = S_n(f;x) + a_nD_n(x)$. Then

$$E_n(f)_{p\theta} \le \|f(\cdot) - S_n(f;\cdot) - a_nD_n(\cdot)\|_{p\theta}.$$

The function $\psi(x) = f(x) - S_n(f;x) - a_nD_n(x)$ belongs to the space $L_{p\theta}[0,1)$, $1 < p < +\infty$, $1 < \theta < +\infty$, and its Fourier–Price coefficients are

$$b_k = \begin{cases} a_n, & k = 0, 1, \ldots, n-1, \\ a_k, & k \ge n. \end{cases}$$

The sequence $b = \{b_k\}_{k=0}^{+\infty}$ belongs to GM; therefore, we can apply Theorem 3.1 to the function ψ. Then

$$E_n(f)_{p\theta} \le \|\psi\|_{p\theta} \le c_{p\theta}\left\{b_0^\theta + \sum_{k=1}^{+\infty} k^{\theta(1-\frac{1}{p})-1} b_k^\theta\right\}^{\frac{1}{\theta}} \le c_{p\theta}'\left\{n^{1-\frac{1}{p}} a_n + \left[\sum_{k=n}^{+\infty} k^{\theta(1-\frac{1}{p})-1} a_k^\theta\right]^{\frac{1}{\theta}}\right\}$$

for $n \in \mathbb{N}$. Thus, the theorem is proved. $\square$

Corollary 3.2. *Let $f \in L[0,1)$ and $f(x) \sim \sum_{k=0}^{+\infty} a_k\varphi_k(x)$ be its Fourier–Price series such that the positive sequence of Fourier–Price coefficients $\{a_k\}_{k=0}^{+\infty}$ belongs to* GM. *If the series $\sum_{k=1}^{+\infty} k^{p-2}a_k^p$ converges for some p, $1 < p < +\infty$, then $f \in L_p[0,1)$, and the following inequality holds*:

$$E_n(f)_p \le c_p\biggl\{n^{1-\frac{1}{p}}\, a_n + \biggl[\sum_{k=n}^{+\infty} k^{p-2}a_k^p\biggr]^{\frac{1}{p}}\biggr\} \qquad \forall\, n \in \mathbb{N}.$$

Here the coefficient $c_p > 0$ depends only on p.

Theorem 3.3. *Let $1 < p < +\infty$, $1 < \theta < +\infty$, $f \in L_{p\theta}[0,1)$, and $\sum_{k=0}^{+\infty} a_k\varphi_k(x)$ be its Fourier–Price series. If the positive sequence of Fourier–Price coefficients $\{a_k\}_{k=0}^{+\infty}$ belongs to* GM, *then the following inequalities hold*:

$$a_{2n} \le c'_{p\theta}\, n^{\frac{1}{p}-1} E_n(f)_{p\theta}, \tag{3.4}$$

$$\biggl\{\sum_{k=n}^{+\infty} k^{\theta(1-\frac{1}{p})-1}a_k^\theta\biggr\}^{\frac{1}{\theta}} \le c_{p\theta} E_{[n/2]}(f)_{p\theta}. \tag{3.5}$$

Here the coefficients $c'_{p\theta}, c_{p\theta} > 0$ depend only on the indicated parameters.

Proof. Using the property of GM sequences of Fourier–Price coefficients and Lemma 2.6, we obtain

$$c_{p\theta}\, n^{\frac{1}{p}} E_n(f)_{p\theta} \ge \biggl|\sum_{k=n}^{2n-1} a_k\biggr| \ge B\, a_{2n}\, n.$$

Thus, $a_{2n} \le c'_{p\theta}\, n^{\frac{1}{p}-1} E_n(f)_{p\theta}$, and inequality (3.4) is established.

Let us prove inequality (3.5). To this end, we apply Theorem 3.1 to the function $g(x) = f(x) - S_n(f;x) - a_nD_n(x)$. Then

$$\biggl\{a_n^\theta \sum_{k=1}^{n-1} k^{\theta(1-\frac{1}{p})-1} + \sum_{k=n}^{+\infty} k^{\theta(1-\frac{1}{p})-1}a_k^\theta\biggr\}^{\frac{1}{\theta}} \le B_{p\theta}\bigl\{\|f - S_n(f)\|_{p\theta} + a_n\|D_n\|_{p\theta}\bigr\}$$

$$\le B'_{p\theta}\Bigl\{E_n(f)_{p\theta} + a_n\, n^{1-\frac{1}{p}}\Bigr\} \le B''_{p\theta}\bigl\{E_n(f)_{p\theta} + E_{[n/2]}(f)_p\bigr\} \le B''_{p\theta}\, E_{[n/2]}(f)_{p\theta}$$

(in the second inequality of the chain we used Lemmas 2.5 and 2.3 and inequality (3.4)). Hence it follows that

$$\biggl\{\sum_{k=n}^{+\infty} k^{\theta(1-\frac{1}{p})-1}a_k^\theta\biggr\}^{\frac{1}{\theta}} \le B''_{p\theta} E_{[n/2]}(f)_{p\theta} \qquad \forall\, n \in \mathbb{N}. \quad \square$$

Corollary 3.3. *Let $f \in L_p[0,1)$, $1 < p < +\infty$, and $\sum_{k=0}^{+\infty} a_k\varphi_k(x)$ be its Fourier–Price series. If the positive sequence of Fourier–Price coefficients $\{a_k\}_{k=0}^{+\infty}$ belongs to* GM, *then the following inequalities hold*:

$$a_{2n} \le c'_p\, n^{\frac{1}{p}-1} E_n(f)_p, \qquad \biggl\{\sum_{k=n}^{+\infty} k^{p-2}a_k^p\biggr\}^{\frac{1}{p}} \le c_p\, E_{[n/2]}(f)_p.$$

Here the coefficients $c_p, c'_p > 0$ depend only on p.

Theorem 3.4. *Let $1 < q < +\infty$, $1 < \tau < +\infty$, and $f \in L_{q\tau}[0,1)$. If the positive sequence of Fourier–Price coefficients $\{a_k\}_{k=0}^{+\infty}$ of f belongs to* GM, *then the following inequalities hold for any p, $1 < p < q$, and $\theta \in (1,+\infty)$:*

$$\left\{\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1} E_k^{\tau}(f)_{p\theta}\right\}^{\frac{1}{\tau}} \le c_{pq\theta\tau}\|f\|_{q\tau}, \tag{3.6}$$

$$n^{\frac{1}{p}-\frac{1}{q}} E_n(f)_{p\theta} + \left[\sum_{k=n}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1} E_k^{\tau}(f)_{p\theta}\right]^{\frac{1}{\tau}} \le c'_{pq\theta\tau}\, E_{[n/2]}(f)_{q\tau}. \tag{3.7}$$

Here the coefficients $c_{pq\theta\tau}, c'_{pq\theta\tau} > 0$ depend only on the indicated parameters.

Proof. Using Theorem 3.2 and the inequality $(|b|+|c|)^{\tau} \le 2^{\tau-1}(|b|^{\tau}+|c|^{\tau})$, $1 < \tau < +\infty$, we have

$$\begin{aligned}\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1} E_k^{\tau}(f)_{p\theta} &\le c_{p\theta}\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left\{k^{1-\frac{1}{p}}\,a_k + \left(\sum_{\nu=k}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}a_\nu^{\theta}\right)^{\frac{1}{\theta}}\right\}^{\tau}\\
&\le c_{p\theta\tau}^{\tau}\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})+\tau(1-\frac{1}{p})-1}a_k^{\tau} + c_{p\theta\tau}^{\tau}\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}a_\nu^{\theta}\right)^{\frac{\tau}{\theta}}\\
&= A_{p\theta\tau}\left\{\sum_{k=1}^{+\infty} k^{\tau(1-\frac{1}{q})-1}a_k^{\tau} + \sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}a_\nu^{\theta}\right)^{\frac{\tau}{\theta}}\right\}\\
&\le A'_{p\theta\tau}\left\{\|f\|_{q\tau}^{\tau} + \sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}a_\nu^{\theta}\right)^{\frac{\tau}{\theta}}\right\}\end{aligned} \tag{3.8}$$

(in the last inequality, we used Theorem 3.1). Let $\frac{\tau}{\theta} > 1$ and $r > \frac{1}{p} - \frac{1}{q}$. Then, applying Lemma 2.7, we obtain

$$\begin{aligned}\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}a_\nu^{\theta}\right)^{\frac{\tau}{\theta}} &\le \sum_{k=1}^{+\infty} k^{-\tau(r-\frac{1}{p}+\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty}\nu^{\theta(r+1-\frac{1}{p})-1}a_\nu^{\theta}\right)^{\frac{\tau}{\theta}}\\
\le B_{\tau\theta}\sum_{k=1}^{+\infty} k^{-\tau(r-\frac{1}{p}+\frac{1}{q})-1}\left(k\cdot k^{\theta(r+1-\frac{1}{p})-1}a_k^{\theta}\right)^{\frac{\tau}{\theta}} &= B_{\tau\theta}\sum_{k=1}^{+\infty} k^{\tau(1-\frac{1}{q})-1}a_\nu^{\tau} \le B'_{\tau q\theta}\|f\|_{q\tau}^{\tau}\end{aligned} \tag{3.9}$$

(in the last inequality, we used Theorem 3.1).

Now, let $\tau = \theta$. Then, changing the order of summation, we obtain

$$\begin{aligned}\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\sum_{\nu=k}^{+\infty}\nu^{\tau(1-\frac{1}{p})-1}a_\nu^{\tau} &= \sum_{\nu=1}^{+\infty}\nu^{\tau(1-\frac{1}{p})-1}a_\nu^{\tau}\sum_{k=1}^{\nu}k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\\
&\le c_{pq\tau}\sum_{\nu=1}^{+\infty}\nu^{\tau(1-\frac{1}{p})-1}a_\nu^{\tau}\,\nu^{\tau(\frac{1}{p}-\frac{1}{q})} = c_{pq\tau}\sum_{\nu=1}^{+\infty}\nu^{\tau(1-\frac{1}{q})-1}a_\nu^{\tau} \le c'_{pq\tau}\|f\|_{q\tau}^{\tau}.\end{aligned} \tag{3.10}$$

If $\frac{\tau}{\theta} < 1$, then

$$\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}\,a_\nu^{\theta}\right)^{\frac{\tau}{\theta}} \le c_{pq\tau}\sum_{k=0}^{+\infty}2^{k\tau(\frac{1}{p}-\frac{1}{q})}\left(\sum_{\nu=2^k}^{+\infty}\nu^{\theta(1-\frac{1}{p})-1}\,a_\nu^{\theta}\right)^{\frac{\tau}{\theta}}. \tag{3.11}$$

Let us consider separately the inner sum:

$$\sum_{\nu=2^k}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1}\, a_\nu^\theta = \sum_{s=k}^{+\infty}\sum_{\nu=2^s}^{2^{s+1}-1} \nu^{\theta(1-\frac{1}{p})-1}\, a_\nu^\theta \le c_{p\theta}^\theta \sum_{s=k}^{+\infty} a_{2^s}^\theta\, 2^{s(1-\frac{1}{p})\theta},$$

since $a_\nu \le C a_{2^s}$ for all $\nu \in [2^s, 2^{s+1}] \cap \mathbb{N}$ if $\{a_\nu\} \in \mathrm{GM}$. Now, taking into account the fact that $\frac{\tau}{\theta} < 1$, we continue inequality (3.11) as follows:

$$\begin{aligned}\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1} a_\nu^\theta\right)^{\frac{\tau}{\theta}} &\le c_{pq\theta\tau}\sum_{k=0}^{+\infty} 2^{k\tau(\frac{1}{p}-\frac{1}{q})}\left[\sum_{s=k}^{+\infty} a_{2^s}^\theta \cdot 2^{s(1-\frac{1}{p})\theta}\right]^{\frac{\tau}{\theta}}\\ &\le c_{pq\theta\tau}\sum_{k=0}^{+\infty} 2^{k\tau(\frac{1}{p}-\frac{1}{q})}\sum_{s=k}^{+\infty} a_{2^s}^\tau \cdot 2^{\tau(1-\frac{1}{p})} = c_{pq\theta\tau}\sum_{s=0}^{+\infty} a_{2^s}^\tau \cdot 2^{\tau(1-\frac{1}{p})}\sum_{k=0}^{s} 2^{k\tau(\frac{1}{p}-\frac{1}{q})}\\ &\le c'_{pq\theta\tau}\sum_{s=0}^{+\infty} a_{2^s}^\tau \cdot 2^{\tau(1-\frac{1}{p})}\cdot 2^{s\tau(\frac{1}{p}-\frac{1}{q})} = c'_{pq\theta\tau}\sum_{s=0}^{+\infty} 2^{s\tau(1-\frac{1}{q})}\, a_{2^s}^\tau.\end{aligned}$$

Since $\{a_k\} \in \mathrm{GM}$ by the hypothesis, we have

$$2^{s\tau(1-\frac{1}{q})}\, a_{2^s}^\tau \le D_{q\tau}\sum_{\nu=2^{s-1}}^{2^s-1} \nu^{\tau(1-\frac{1}{q})-1} a_\nu^\tau \qquad \forall\, s \in \mathbb{N}.$$

Therefore,

$$\sum_{s=0}^{+\infty} 2^{s\tau(1-\frac{1}{q})}\, a_{2^s}^\tau \le D'_{q\tau}\left\{a_1^\tau + \sum_{s=1}^{+\infty}\sum_{\nu=2^{s-1}}^{2^s-1} \nu^{\tau(1-\frac{1}{q})-1} a_\nu^\tau\right\} \le D''_{q\tau}\sum_{\nu=1}^{+\infty} \nu^{\tau(1-\frac{1}{q})-1} a_\nu^\tau.$$

Thus, for $\frac{\tau}{\theta} < 1$, the continuation of inequality (3.11) ends with the chain

$$\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}\left(\sum_{\nu=k}^{+\infty} \nu^{\theta(1-\frac{1}{p})-1}\, a_\nu^\theta\right)^{\frac{\tau}{\theta}} \le D_{pq\theta\tau}\sum_{\nu=1}^{+\infty} \nu^{\tau(1-\frac{1}{q})-1}\, a_\nu^\tau \le B_{pq\theta\tau}\|f\|_{q\tau}^\tau. \tag{3.12}$$

Now, from (3.8)–(3.10) and (3.12) we obtain

$$\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1} E_k^\tau(f)_{p\theta} \le c_{pq\theta\tau}\|f\|_{q\tau}^\tau.$$

To prove the second inequality formulated in the theorem, consider the auxiliary function

$$g(x) = f(x) - S_n(f;x) - a_n D_n(x).$$

Since $f \in L_{q\tau}[0,1)$, it is clear that $g \in L_{q\tau}[0,1)$. Moreover,

$$E_k(g)_{p\theta} = \begin{cases} E_k(f)_{p\theta} & \text{for } k \ge n,\\ E_n(f)_{p\theta} & \text{for } 0 \le k \le n-1.\end{cases}$$

Keeping this fact in mind, we apply inequality (3.6) to the function g; then

$$n^{(\frac{1}{p}-\frac{1}{q})\tau} E_n^\tau(f)_{p\theta} + \sum_{k=n}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1} E_k^\tau(f)_{p\theta} \le A_{pq\theta\tau}^\tau\|g\|_{q\tau}^\tau. \tag{3.13}$$

Next, applying Lemmas 2.5, 2.3 and Theorem 3.3, we obtain

$$\|g\|_{q\tau} \le \|f - S_n(f)\|_{q\tau} + a_n\|D_n\|_{q\tau} \le c_{q\tau}\big(E_n(f)_{q\tau} + a_n n^{1-\frac{1}{q}}\big)$$
$$\le c'_{q\tau}\big(E_n(f)_{q\tau} + E_{[n/2]}(f)_{q\tau}\big) \le c''_{q\tau}E_{[n/2]}(f)_{q\tau}. \tag{3.14}$$

Now, (3.13) and (3.14) imply the desired inequality (3.7). □

Corollary 3.4. *Let $1 < q < +\infty$ and $f \in L_q[0,1)$. If the Fourier–Price coefficients $\{a_k\}_{k=0}^{+\infty}$ of the function f belong to* GM, *then the following inequalities are valid for any p, $1 < p < q$:*

$$\left\{\sum_{k=1}^{+\infty} k^{\frac{q}{p}-2}E_k^q(f)_p\right\}^{\frac{1}{q}} \le c_{pq}\|f\|_q, \qquad n^{\frac{1}{p}-\frac{1}{q}}E_n(f)_q + \left[\sum_{k=n}^{+\infty} k^{\frac{q}{p}-2}E_k^q(f)_p\right]^{\frac{1}{q}} \le c'_{pq}E_{[n/2]}(f)_q.$$

Theorem 3.5. *Let $1 < p < +\infty$, $1 < \theta < +\infty$, $f \in L_{p\theta}[0,1)$, $p < q < +\infty$, and $1 < \tau < +\infty$. If the positive sequence of Fourier–Price coefficients $\{a_k\}_{k=0}^{+\infty}$ of f belongs to* GM, *then $f \in L_{q\tau}[0,1)$ if and only if the series*

$$\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}E_k^\tau(f)_{p\theta}$$

converges. In this case, the following inequalities hold:

$$c'_{pq\theta\tau}\left[\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}E_k^\tau(f)_{p\theta}\right]^{\frac{1}{\tau}} \le \|f\|_{q\tau} \le c_{pq\theta\tau}\left\{\|f\|_{p\theta} + \left[\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1}E_k^\tau(f)_{p\theta}\right]^{\frac{1}{\tau}}\right\}.$$

Here the coefficients $c_{pq\theta\tau}, c'_{pq\theta\tau} > 0$ depend only on the indicated parameters.

Proof. The necessity is established in Theorem 3.4. Let us prove the sufficiency. By Theorem 3.1, we have

$$\|S_{2^n}(f)\|_{q\tau} \le c_{q\tau}\left\{a_0^\tau + \sum_{k=1}^{2^n-1} k^{\tau(1-\frac{1}{q})-1}a_k^\tau\right\}^{\frac{1}{\tau}}. \tag{3.15}$$

It is clear that

$$a_0 \le c_{p\theta}\|f\|_{p\theta}, \qquad a_1 \le c_{p\theta}\|f\|_{p\theta}. \tag{3.16}$$

Next, since $\{a_k\}_{k=1}^{+\infty} \in \mathrm{GM}$, we obtain

$$\sum_{k=1}^{2^n-1} k^{\tau(1-\frac{1}{q})-1}a_k^\tau = \sum_{\nu=0}^{n-1}\sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\tau(1-\frac{1}{q})-1}a_k^\tau \le c_{\tau q}\sum_{\nu=0}^{n-1} 2^{\nu(1-\frac{1}{q})\tau}a_{2^\nu}^\tau$$
$$\le c'_{pq\theta\tau}\left\{\sum_{\nu=1}^{n-1} 2^{\nu(1-\frac{1}{q})}\left(2^{(\nu-1)(\frac{1}{p}-1)}E_{2^{\nu-1}}(f)_{p\theta}\right)^\tau + \|f\|_{p\theta}^\tau\right\} = c''_{pq\theta\tau}\left\{\|f\|_{p\theta}^\tau + \sum_{s=0}^{n-2} 2^{s(\frac{1}{p}-\frac{1}{q})\tau}E_{2^s}^\tau(f)_{p\theta}\right\}$$

(when passing to the second row, we used Theorem 3.3 and inequality (3.16)). Thus, taking into account this inequality and (3.16), we can continue inequality (3.15):

$$\|S_{2^n}(f)\|_{q\tau} \le A_{pq\theta\tau}\left\{\|f\|_{p\theta} + \left[\sum_{s=0}^{n-2} 2^{s(\frac{1}{p}-\frac{1}{q})\tau}E_{2^s}^\tau(f)_{p\theta}\right]^{\frac{1}{\tau}}\right\} \qquad \forall\, n \in \mathbb{N}.$$

The convergence of the series $\sum_{s=0}^{+\infty} 2^{s\tau(\frac{1}{p}-\frac{1}{q})} E_{2^s}^{\tau}(f)_{p\theta}$ and Lemma 2.3 imply that

$$\|f\|_{q\tau} \le A_{pq\theta\tau}\left\{\|f\|_{p\theta} + \left[\sum_{s=0}^{+\infty} 2^{s\tau(\frac{1}{p}-\frac{1}{q})} E_{2^s}^{\tau}(f)_{p\theta}\right]^{\frac{1}{\tau}}\right\} \le A'_{pq\theta\tau}\left\{\|f\|_{p\theta} + \left[\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1} E_{k}^{\tau}(f)_{p\theta}\right]^{\frac{1}{\tau}}\right\}.$$

The theorem is proved. □

Remark 3.2. Theorem 3.5 is presented for completeness. The fact that the convergence of the series

$$\sum_{k=1}^{+\infty} k^{\tau(\frac{1}{p}-\frac{1}{q})-1} E_{k}^{\tau}(f)_{p\theta}, \qquad 1 < p < q < +\infty,$$

for f guarantees the inclusion $f \in L_{q\tau}[0,1)$ was established in [23] without the condition of monotonicity of its sequence of Fourier–Price coefficients. For the first time, a similar result was established by Ul'yanov [29] in terms of the best trigonometric approximations in Lebesgue spaces. Later this area was developed by M.F. Timan, M.K. Potapov, B.I. Golubov, E.A. Storozhenko, V.A. Andrienko, V.I. Kolyada, N. Temirgaliev, E.S. Smailov, G.A. Akishev, and others.

4. HARDY–LITTLEWOOD TYPE THEOREM IN NIKOL'SKII–BESOV SPACES WITH THE PRICE BASIS

Theorem 4.1. *Let $f \in L_1[0,1)$ and $f(x) \sim \sum_{k=0}^{+\infty} a_k\varphi_k(x)$ be its Fourier–Price series. Suppose that the positive sequence of Fourier–Price coefficients $\{a_k\}_{k=0}^{+\infty}$ belongs to* GM. *Then $f \in B_{p\theta}^r(\varphi;[0,1))$, where $r > 0$, $1 < p < +\infty$, and $1 \le \theta \le +\infty$, if and only if the quantity*

$$\left\{\sum_{k=1}^{+\infty} k^{r\theta+\theta-\frac{\theta}{p}-1} a_k^{\theta}\right\}^{\frac{1}{\theta}} \tag{4.1}$$

be finite. In this case, the following inequalities are valid:

$$c'_{p\theta}\left\{a_0^{\theta} + \sum_{k=1}^{+\infty} k^{r\theta+\theta-\frac{\theta}{p}-1} a_k^{\theta}\right\}^{\frac{1}{\theta}} \le |f; B_{p\theta}^r(\varphi;[0,1))| \le c_{p\theta}\left\{a_0^{\theta} + \sum_{k=1}^{+\infty} k^{r\theta+\theta-\frac{\theta}{p}-1} a_k^{\theta}\right\}^{\frac{1}{\theta}}.$$

Here the coefficients $c_{p\theta}, c'_{p\theta} > 0$ depend only on the indicated parameters.

Proof. I. First, consider the case of $1 \le \theta < +\infty$.

Sufficiency. Let the series (4.1) converge. First, we show that under the hypotheses of the theorem $f \in L_p[0,1)$. To this end, in view of $\{a_k\}_{k=0}^{+\infty} \in$ GM, it suffices to establish the convergence of the series $a_0^p + \sum_{k=1}^{+\infty} k^{p-2} a_k^p$.

Let, first, $\frac{\theta}{p} > 1$. Applying Hölder's inequality for $\lambda = \frac{\theta}{p}$ and $\lambda' = \frac{\theta}{\theta-p}$, we obtain

$$a_0^p + \sum_{k=1}^{N} k^{p-2} a_k = a_0^p + \sum_{k=1}^{N} k^{p-1+rp-\frac{p}{\theta}}\, k^{\frac{p}{\theta}-rp-1} a_k$$
$$\le a_0^p + \left\{\sum_{k=1}^{N} k^{\theta-\frac{\theta}{p}+r\theta-1} a_k^{\theta}\right\}^{\frac{p}{\theta}} \left\{\sum_{k=1}^{N} k^{-1-\frac{r\theta p}{\theta-p}}\right\}^{\frac{\theta-p}{\theta}} \le c_{p\theta r}\left\{a_0^{\theta} + \sum_{k=1}^{+\infty} k^{\theta-\frac{\theta}{p}+r\theta-1} a_k^{\theta}\right\}^{\frac{p}{\theta}} < +\infty$$

for $N \in \mathbb{N}$.

In the case of $\theta = p$, we have $(r\theta + \theta - \frac{\theta}{p} - 1 = rp + p - 2)$

$$a_0^p + \sum_{k=1}^{N} k^{p-2} a_k^p \le a_0^p + \sum_{k=1}^{N} k^{rp+p-2} a_k^p \le a_0^p + \sum_{k=1}^{+\infty} k^{rp+p-2} a_k^p < +\infty \qquad \forall\, N \in \mathbb{N}.$$

Consider the case of $\frac{\theta}{p} < 1$:

$$\left\{a_0^p + \sum_{l=0}^{N} 2^{l(p-1)} a_{2^l}^p\right\}^{\frac{\theta}{p}} \le a_0^\theta + \sum_{l=0}^{N} 2^{l\left(\theta - \frac{\theta}{p}\right)} a_{2^l}^\theta \le a_0^\theta + \sum_{l=0}^{+\infty} 2^{l\left(r\theta + \theta - \frac{\theta}{p}\right)} a_{2^l}^\theta < +\infty \qquad \forall\, N \in \mathbb{N}.$$

Hence, according to Corollary 3.1, we can state that $f \in L_p[0,1)$, $1 < p < +\infty$, and, moreover,

$$\|f\|_p \le c_p \left\{a_0^p + \sum_{k=1}^{+\infty} k^{p-2} a_k^p\right\}^{\frac{1}{p}} \le c_{p\theta r}\left\{a_0^\theta + \sum_{k=1}^{+\infty} k^{\theta - \frac{\theta}{p} + \theta r - 1} a_k^\theta\right\}^{\frac{1}{\theta}} < +\infty.$$

Now, let us estimate the seminorm of the Nikol'skii–Besov space with the Price basis.

First, consider the case of $\frac{\theta}{p} \ge 1$. Since the series $\sum_{k=1}^{+\infty} k^{p-2} a_k^p$ converges, in view of Corollary 3.2 we have

$$\begin{aligned}
\|f\|_{b_{p\theta}^r}^\theta &= \sum_{\nu=0}^{+\infty} 2^{\nu r\theta} E_{2^\nu}^\theta(f)_p \le c_p \cdot 2^{\theta-1}\left\{\sum_{\nu=0}^{+\infty} 2^{\nu r\theta} \cdot 2^{\nu\theta\left(1-\frac{1}{p}\right)} a_{2^\nu}^\theta + \sum_{\nu=0}^{+\infty} 2^{\nu r\theta}\left[\sum_{k=2^\nu+1}^{+\infty} a_k^p\, k^{p-2}\right]^{\frac{\theta}{p}}\right\} \\
&\le c_{p\theta}\left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta\left(r+1-\frac{1}{p}\right)} a_{2^\nu}^\theta + \sum_{\nu=0}^{+\infty} 2^{\nu r\theta}\left[\sum_{k=\nu}^{+\infty} a_{2^k}^p \cdot 2^{k(p-1)}\right]^{\frac{\theta}{p}}\right\} \\
&\le c_{p\theta r}\left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta\left(r+1-\frac{1}{p}\right)} a_{2^\nu}^\theta\right\} \le c'_{p\theta r}\left\{\sum_{\nu=1}^{+\infty} a_\nu^\theta\, \nu^{r\theta + \theta - \frac{\theta}{p} - 1}\right\} < +\infty
\end{aligned}$$

(when passing to the third row, we used Lemma 2.8).

Now, let $\frac{\theta}{p} < 1$. In this case we also rely on Corollary 3.2:

$$\begin{aligned}
\|f\|_{b_{p\theta}^r}^\theta &= \sum_{\nu=0}^{+\infty} 2^{\nu r\theta} E_{2^\nu}^\theta(f)_p \le c_{p\theta} \sum_{\nu=0}^{+\infty} 2^{\nu r\theta}\left\{2^{\nu\left(1-\frac{1}{p}\right)} a_{2^\nu} + \left\{\sum_{k=2^\nu+1}^{+\infty} a_k^p\, k^{p-2}\right\}^{\frac{1}{p}}\right\}^\theta \\
&\le c_{p\theta} \cdot 2^{\theta-1}\left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta\left(r+1-\frac{1}{p}\right)} a_{2^\nu}^\theta + \sum_{\nu=0}^{+\infty} 2^{\nu r\theta}\left[\sum_{k=2^\nu+1}^{+\infty} k^{p-2}\, a_k^p\right]^{\frac{\theta}{p}}\right\} \\
&\le c'_{p\theta}\left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta\left(r+1-\frac{1}{p}\right)} a_{2^\nu}^\theta + \sum_{\nu=0}^{+\infty} 2^{\nu r\theta} \sum_{k=\nu}^{+\infty}\left(a_{2^k}^p \cdot 2^{k(p-1)}\right)^{\frac{\theta}{p}}\right\} \\
&= c'_{p\theta}\left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta\left(r+1-\frac{1}{p}\right)} a_{2^\nu}^\theta + \sum_{k=0}^{+\infty} 2^{k\theta\left(1-\frac{1}{p}\right)} a_{2^k}^\theta \sum_{\nu=0}^{k} 2^{\nu r\theta}\right\} \\
&\le c''_{p\theta r}\left\{\sum_{k=0}^{+\infty} 2^{k\theta\left(r+1-\frac{1}{p}\right)} a_{2^k}^\theta\right\} \le c'''_{p\theta r}\left\{\sum_{k=0}^{+\infty} a_k^\theta\, k^{r\theta + \theta - \frac{\theta}{p} - 1}\right\}.
\end{aligned}$$

Thus,

$$|f; B^r_{p\theta}(\varphi;[0,1))| = \|f\|_p + \|f\|_{b^r_{p\theta}} \le \bar{c}_{p\theta r}\left\{a_0^\theta + \sum_{k=1}^{+\infty} a_k^\theta\, k^{\theta(r+1-\frac{1}{p})-1}\right\}^{\frac{1}{\theta}},$$

where $r>0$, $1<p<+\infty$, and $1\le\theta<+\infty$.

Necessity. Let $f\in B^r_{p\theta}[0,1)$, $r>0$, $1<p<+\infty$, and $1\le\theta<+\infty$. The functional $|f; B^r_{p\theta}(\varphi;[0,1))| = \|f\|_p + \|f\|_{b^r_{p\theta}}$ is the norm of a given element in the Nikol'skii–Besov space with the Price basis. Then, by Corollary 3.3, we have

$$\|f\|_{b^r_{p\theta}} = \left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta r} E^\theta_{2^\nu}(f)_p\right\}^{\frac{1}{\theta}} = \left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta r}\left(2^{\nu(\frac{1}{p}-1)}\cdot 2^{\nu(1-\frac{1}{p})} E_{2^\nu}(f)_p\right)^\theta\right\}^{\frac{1}{\theta}}$$

$$\ge A_{p\theta}\left\{\sum_{\nu=0}^{+\infty} 2^{\nu\theta r}\cdot 2^{\nu\theta(1-\frac{1}{p})} a^\theta_{2^{\nu+1}}\right\}^{\frac{1}{\theta}} = A'_{p\theta r}\left\{\sum_{k=1}^{+\infty} 2^{k\theta(r+1-\frac{1}{p})} a^\theta_{2^k}\right\}^{\frac{1}{\theta}}.$$

Moreover, $a_0\le c_{p\theta}\|f\|_p$ and $a_1\le c'_{p\theta}\|f\|_p$.

Hence,

$$|f; B^r_{p\theta}(\varphi;[0,1))| \ge A''_{p\theta r}\left\{a_0^\theta + \sum_{k=0}^{+\infty} 2^{k\theta(r+1-\frac{1}{p})} a^\theta_{2^k}\right\}^{\frac{1}{\theta}} \ge A''_{p\theta r}\left\{a_0^\theta + \sum_{k=1}^{+\infty} k^{\theta(r+1-\frac{1}{p})-1} a_k^\theta\right\}^{\frac{1}{\theta}}.$$

II. Now, consider the case of $\theta=+\infty$.

Sufficiency. In this case, the hypothesis of the theorem says that the quantity

$$\sup\{k^{r+1-\frac{1}{p}} a_k:\ k\in\mathbb{N}\} \tag{4.2}$$

is finite.

Let us show that the series $a_0^p + \sum_{k=1}^{+\infty} k^{p-2} a_k^p$ converges under this condition. Consider the sum

$$\left\{\sum_{k=1}^{N} k^{p-2} a_k^p\right\}^{\frac{1}{p}} = \left\{\sum_{k=1}^{N} k^{p(1-\frac{1}{p}+r)} a_k^p\, k^{-1-rp}\right\}^{\frac{1}{p}}$$

$$\le \sup\{k^{r+1-\frac{1}{p}} a_k:\ k\in\mathbb{N}\}\left\{\sum_{k=1}^{+\infty} k^{-1-rp}\right\}^{\frac{1}{p}} = c_{pr}D_{pr} < +\infty \qquad \forall N\in\mathbb{N}.$$

Therefore, by Corollary 3.1, we have $f\in L_p[0,1)$, $1<p<+\infty$, and

$$\|f\|_p \le c_p\left\{a_0^p + \sum_{k=1}^{+\infty} k^{p-2} a_k^p\right\}^{\frac{1}{p}} \le c_{pr}\sup\{a_0 + k^{r+1-\frac{1}{p}} a_k:\ k\in\mathbb{N}\} < +\infty. \tag{4.3}$$

Now, let us estimate the seminorm of the space $B^r_{p\infty}(\varphi;[0,1))$. Applying first Corollary 3.2 and then inequality (4.3), we obtain

$$\|f\|_{b^r_{p\infty}} = \sup\{2^{\nu r} E_{2^\nu}(f)_p:\ \nu\in\mathbb{Z}^+\} \le c_p \sup\left[2^{\nu r}\left\{2^{\nu(1-\frac{1}{p})} a_{2^\nu} + \left(\sum_{k=2^\nu}^{+\infty} k^{p-2} a_k^p\right)^{\frac{1}{p}}\right\}\right]$$

$$\le c_p\sup\{a_0 + k^{r+1-\frac{1}{p}} a_k:\ k\in\mathbb{N}\}.$$

Thus, if (4.2) is finite, we have

$$|f; B^r_{p\theta}(\varphi;[0,1))| \le A_{rp} \sup\bigl\{a_0 + k^{r+1-\frac{1}{p}} a_k\colon\ k\in\mathbb{N}\bigr\}.$$

Necessity. Let $f \in B^r_{p\infty}(\varphi;[0,1))$, where $r>0$ and $1<p<+\infty$. Applying the second inequality in Corollary 3.3, we obtain

$$\begin{aligned}\|f\|_{b^r_{p\infty}} &\ge \sup\bigl\{2^{\nu r} E_{2^\nu}(f)_p\colon\ \nu\in\mathbb{Z}^+\bigr\} \ge c_p \sup\Biggl\{2^{\nu r}\Biggl(\sum_{k=2^{\nu+1}}^{+\infty} k^{p-2} a_k^p\Biggr)^{\frac{1}{p}}\colon\ \nu\in\mathbb{Z}^+\Biggr\}\\ &= c_p \sup\Biggl\{2^{\nu r}\Biggl(\sum_{k=2^{\nu+1}}^{+\infty} k^{p(1-\frac{1}{p})-1} a_k^p\Biggr)^{\frac{1}{p}}\colon\ \nu\in\mathbb{Z}^+\Biggr\}. \qquad (4.4)\end{aligned}$$

Since the inclusion $2^{\nu+2}\in[k,2k]$ is valid for any $k\in[2^{\nu+1}+1,\dots,2^{\nu+2}-1]$, according to (2.2) we have $c^{-1}a_{2^{\nu+2}} \le a_k$. Taking this fact into account, we obtain

$$\begin{aligned}\Biggl(\sum_{k=2^{\nu+1}}^{+\infty} k^{p(1-\frac{1}{p})-1} a_k^p\Biggr)^{\frac{1}{p}} &\ge \Biggl(\sum_{k=2^{\nu+1}+1}^{2^{\nu+2}} k^{p(1-\frac{1}{p})-1} a_k^p\Biggr)^{\frac{1}{p}} \ge c^{-1} a_{2^{\nu+2}} \Biggl(\sum_{k=2^{\nu+1}+1}^{2^{\nu+2}} k^{p(1-\frac{1}{p})-1}\Biggr)^{\frac{1}{p}}\\ &= D_{pr}\, a_{2^{\nu+2}}\cdot 2^{(\nu+1)(1-\frac{1}{p})} = D'_{pr}\, a_{2^{\nu+2}}\cdot 2^{(\nu+3)(1-\frac{1}{p})}\\ &= D'_{pr}\, a_{2^{\nu+2}} \max\bigl\{k^{1-\frac{1}{p}}\colon\ k\in[2^{\nu+2}+1, 2^{\nu+3}]\bigr\}\\ &\ge c^{-1} D'_{pr} \max\bigl\{a_k k^{1-\frac{1}{p}}\colon\ k\in[2^{\nu+2}+1, 2^{\nu+3}]\bigr\} \qquad (4.5)\end{aligned}$$

(in the last inequality, we used the property (2.2) of the sequence $\bar a\in \mathrm{GM}$). Now, combining (4.4) and (4.5), we have

$$\begin{aligned}\|f\|_{b^r_{p\infty}} &\ge D''_{pr} \sup\Bigl\{2^{\nu r} \max\bigl\{a_k k^{1-\frac{1}{p}}\colon\ k\in[2^{\nu+2}+1, 2^{\nu+3}]\bigr\}\colon\ \nu\in\mathbb{Z}^+\Bigr\}\\ &= 2^{-3r} D''_{pr} \sup\Bigl\{2^{(\nu+3)r} \max\bigl\{a_k k^{1-\frac{1}{p}}\colon\ k\in[2^{\nu+2}+1, 2^{\nu+3}]\bigr\}\colon\ \nu\in\mathbb{Z}^+\Bigr\}\\ &\ge D'''_{pr} \sup\Bigl\{\max\bigl\{a_k k^{r+1-\frac{1}{p}}\colon\ k\in[2^{\nu+2}+1, 2^{\nu+3}]\bigr\}\colon\ \nu\in\mathbb{Z}^+\Bigr\}\\ &= D'''_{pr} \sup\bigl\{a_k k^{r+1-\frac{1}{p}}\colon\ k\in\mathbb{N},\ k\ge 5\bigr\} \ge D^{(\mathrm{iv})}_{pr} \sup\bigl\{a_k k^{r+1-\frac{1}{p}}\colon\ k\in\mathbb{N}\bigr\}.\end{aligned}$$

Thus,

$$|f; B^r_{p\infty}(\varphi;[0,1))| \ge \sup\bigl\{2^{\nu r} E_{2^\nu}(f)_p\colon\ \nu\in\mathbb{Z}^+\bigr\} \ge D^{(\mathrm{iv})}_{pr} \sup\bigl\{k^{r+1-\frac{1}{p}} a_k\colon\ k\in\mathbb{N}\bigr\}.$$

It was clear during the proof that the coefficient $D^{(\mathrm{iv})}_{pr} > 0$ depends only on the indicated parameters. □

REFERENCES

1. N. Yu. Agafonova, "On the best approximation of functions with respect to multiplicative systems and the properties of their Fourier coefficients," Anal. Math. **33**, 247–262 (2007).
2. J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction* (Springer, Berlin, 1976).
3. N. A. Bokaev and E. S. Smailov, *Fourier Series with Respect to Multiplicative Systems: A Textbook* (Karagand. Gos. Univ., Karaganda, 1993) [in Russian].

4. E. A. Borisova, “On embedding of classes of functions defined by sequences of best approximations with respect to some orthonormal systems,” Cand. Sci. (Phys.–Math.) Dissertation (Moscow State Univ., Moscow, 1988).
5. M. I. D’yachenko, “The Hardy–Littlewood theorem for trigonometric series with generalized monotone coefficients,” Izv. Vyssh. Uchebn. Zaved., Mat., No. 5, 38–47 (2008) [Russ. Math. **52** (5), 32–40 (2008)].
6. M. I. Dyachenko and E. D. Nursultanov, “Hardy–Littlewood theorem for trigonometric series with α-monotone coefficients,” Mat. Sb. **200** (11), 45–60 (2009) [Sb. Math. **200**, 1617–1631 (2009)].
7. B. I. Golubov, A. V. Efimov, and V. A. Skvortsov, *Walsh Series and Transforms: Theory and Applications* (Nauka, Moscow, 1987; Kluwer, Dordrecht, 1991).
8. M. L. Gol’dman, “On imbedding Nikol’skii–Besov spaces with moduli of continuity of general form into Lorentz spaces,” Dokl. Akad. Nauk SSSR **277** (1), 20–24 (1984) [Sov. Math., Dokl. **30**, 11–16 (1984)].
9. A. B. Gulisashvili, “Distribution functions and trigonometric series with monotonically decreasing coefficients,” Mat. Zametki **10** (1), 3–10 (1971) [Math. Notes **10**, 427–430 (1971)].
10. A. B. Gulisashvili, V. A. Rodin, and E. M. Semenov, “Fourier coefficients of summable functions,” Mat. Sb. **102** (3), 362–371 (1977) [Math. USSR, Sb. **31**, 319–328 (1977)].
11. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge Univ. Press, Cambridge, 1934).
12. V. M. Kokilashvili, “On approximation of periodic functions,” Tr. Tbilis. Mat. Inst. Razmadze **34**, 51–81 (1968).
13. A. A. Konyushkov, “Best approximations by trigonometric polynomials and Fourier coefficients,” Mat. Sb. **44** (1), 53–84 (1958).
14. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* (Nauka, Moscow, 1978; Am. Math. Soc., Providence, RI, 1982), Transl. Math. Monogr. **54**.
15. L. Leindler, “Best approximation and Fourier coefficients,” Anal. Math. **31**, 117–129 (2005).
16. E. D. Nursultanov, “Net spaces and inequalities of Hardy–Littlewood type,” Mat. Sb. **189** (3), 83–102 (1998) [Sb. Math. **189**, 399–419 (1998)].
17. E. D. Nursultanov, “On the coefficients of multiple Fourier series in L_p-spaces,” Izv. Ross. Akad. Nauk, Ser. Mat. **64** (1), 95–122 (2000) [Izv. Math. **64**, 93–120 (2000)].
18. V. A. Rodin, “The Hardy–Littlewood theorem for the cosine series in a symmetric space,” Mat. Zametki **20** (2), 241–246 (1976) [Math. Notes **20**, 693–696 (1976)].
19. V. A. Rodin, “On membership of the sum of a cosine series with monotone coefficients in a symmetric space,” Izv. Vyssh. Uchebn. Zaved., Mat., No. 8, 60–64 (1979) [Sov. Math. **23** (8), 61–65 (1979)].
20. Y. Sagher, “An application of interpolation theory to Fourier series,” Stud. Math. **41** (2), 169–181 (1972).
21. E. M. Semenov, “Interpolation of linear operators and estimates of Fourier coefficients,” Dokl. Akad. Nauk SSSR **176** (6), 1251–1254 (1967) [Sov. Math., Dokl. **8**, 1315–1319 (1967)].
22. E. S. Smailov and A. U. Bimendina, “Hardy–Littlewood theorem for Fourier–Price series with quasimonotone coefficients in the Lorentz space,” Vestn. Karagand. Univ., Mat., No. 2, 3–9 (2005).
23. E. S. Smailov and A. U. Bimendina, “On the embedding in the Lorentz space,” Vestn. Karagand. Univ., Mat., No. 2, 75–83 (2008).
24. E. S. Smailov and Z. R. Suleimenova, “Embedding theorems for Besov spaces over multiplicative Price bases,” Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk **243**, 313–319 (2003) [Proc. Steklov Inst. Math. **243**, 302–308 (2003)].
25. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton Univ. Press, Princeton, NJ, 1971).
26. S. Tazabekov and E. S. Smailov, *Trigonometric Fourier Series with Quasimonotone Coefficients*, Available from VINITI, No. 5255-V88 (Moscow, 1988).
27. S. Tikhonov, “Trigonometric series with general monotone coefficients,” Math. Anal. Appl. **326**, 721–735 (2007).
28. M. F. Timan and K. Tukhliev, “Properties of certain orthonormal systems,” Izv. Vyssh. Uchebn. Zaved., Mat., No. 9, 65–73 (1983) [Sov. Math. **27** (9), 74–84 (1983)].
29. P. L. Ul’yanov, “Imbedding theorems and relations between best approximations (moduli of continuity) in different metrics,” Mat. Sb **81** (1), 104–131 (1970) [Math. USSR, Sb. **10**, 103–126 (1970)].
30. W.-S. Yong, “Mean convergence of generalized Walsh–Fourier series,” Trans. Am. Math. Soc. **218**, 311–320 (1976).
31. A. Zygmund, *Trigonometric Series* (Cambridge Univ. Press, Cambridge, 1959), Vol. 2.

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