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Estimations of the best M -term approximations of functions in the Lorentz space with constructive methods

This paper considers the Lorentz space of periodic functions of many variables with the anisotropic norm, of functional Nikol'skii-Besov's class and of the best M -term approximation of function. We have established sufficient conditions for the function to belong to one of the Lorentz spaces in another. We obtain upper and lower bounds for the best M -member approximations of functions from the Nikol'skii-Besov class in the anisotropic Lorentz space To prove the upper bound, we used a new constructive method developed by V.N. Temlyakov.

Keywords: Lorentz space, Nikol'ski-Besov class, the best M -term approximations, approximation, sufficient conditions, estimate.

Introduction

Let $\bar{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$, $I^m = [0, 2\pi]^m$ and numbers $\theta_j, q_j \in [1, +\infty)$, $j = 1, \dots, m$. Let $L_{\bar{q}, \bar{\theta}}^*(I^m)$ denotes the space of Lebesgue measurable functions $f(\bar{x})$ defined on \mathbb{R}^m with the period 2π with respect to each variable such that the quantity

$$\|f\|_{\bar{q}, \bar{\theta}}^* = \left[\int_0^{2\pi} \frac{\theta_m - 1}{t_m^{\theta_m - 1}} \left[\dots \left[\int_0^{2\pi} \left(f^{*1, \dots, *m}(t_1, \dots, t_m) \right)^{\theta_1} t_1^{\theta_1 - 1} dt_1 \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} dt_m \right]^{\frac{1}{\theta_m}},$$

is finite, where $f^{*1, \dots, *m}(t_1, \dots, t_m)$ is a non-increasing rearrangement of the function $|f(\bar{x})|$ in each variable x_j , whereas the other variables are fixed [1].

In case when the $q_1 = \dots = q_m = \theta_1 = \dots = \theta_m = q$, the space of Lorentz $L_{\bar{q}, \bar{\theta}}^*(I^m)$ coincides with the space of Lebesgue $L_q(I^m)$ with the norm ([2], Ch. I, item 1.1)

$$\|f\|_q = \left[\int_0^{2\pi} \dots \int_0^{2\pi} |f(x_1, \dots, x_m)|^q dx_1 \dots dx_m \right]^{\frac{1}{q}}.$$

Let $\overset{\circ}{L}_{\bar{q}, \bar{\theta}}^*(I^m)$ be the set of all functions $f \in L_{\bar{q}, \bar{\theta}}^*(I^m)$ such that

$$\int_0^{2\pi} f(\bar{x}) dx_j = 0, \quad \forall j = 1, \dots, m.$$

For any function $f \in L_1(I^m) = L(I^m)$, let

$$\sum_{\bar{n} \in \mathbb{Z}^m} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle}$$

be function's Fourier series with respect to the multiple trigonometric system $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}_{\mathbb{Z}^m}$, where \mathbb{Z}^m is the set of points in \mathbb{R}^m with integer coordinates.

Suppose

$$\delta_{\bar{s}}(f, \bar{x}) = \sum_{\bar{n} \in \rho(\bar{s})} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

where $\langle \bar{y}, \bar{x} \rangle = \sum_{j=1}^m y_j x_j$, $s_j = 1, 2, \dots$ and

$$\rho(\bar{s}) = \{ \bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : 2^{s_j-1} \leq |k_j| < 2^{s_j}, j = 1, \dots, m \}.$$

For a number sequence, we will write $\{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m} \in l_{\bar{p}}$ if

$$\| \{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m} \|_{l_{\bar{p}}} = \left\{ \sum_{n_m=-\infty}^{\infty} \left[\dots \left[\sum_{n_1=-\infty}^{\infty} |a_{\bar{n}}|^{p_1} \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} \right\}^{\frac{1}{p_m}} < +\infty,$$

where $\bar{p} = (p_1, \dots, p_m)$, $1 \leq p_j < +\infty$, $j = 1, 2, \dots, m$.

The spaces $S_p^{\bar{r}}H$ and $S_{p,\theta}^{\bar{r}}B$ of functions with the dominating mixed derivative were introduced by S.M. Nikol'skii [3] and T.I. Amanov ([4] Ch.I, item 17). The spaces $S_p^{\bar{r}}H$, $S_{p,\theta}^{\bar{r}}B$ are called Nikolski-Besov's space, or, sometimes, Nikolski-Besov-Amanov's space.

P. I. Lizorkin and S. M. Nikol'skii [5] investigated a decomposition of elements of the space $S_{p,\theta}^{\bar{r}}B$. We will use their definition.

Let $\bar{r} = (r_1, \dots, r_m)$, $r_j > 0, j = 1, \dots, m$, $1 \leq p, \theta \leq +\infty$. Suppose $S_{p,\theta}^{\bar{r}}B$ is the space all of functions $f \in L_{\bar{q},\bar{\theta}}^*(I^m)$ such that

$$\|f\|_{S_{p,\theta}^{\bar{r}}B} = \left[\int_0^{2\pi} \dots \int_0^{2\pi} \|\Delta_{\bar{t}}^{\bar{k}} f(\bullet)\|_p^\theta \prod_{j=1}^m \frac{dt_j}{t_j^{1+\theta r_j}} \right]^{\frac{1}{\theta}} < +\infty,$$

where $\Delta_{\bar{t}}^{\bar{k}} f(\bar{x}) = \Delta_{t_m}^{k_m} (\dots \Delta_{t_1}^{k_1} f(\bar{x}))$ is the mixed difference of order \bar{k} with step $\bar{t} = (t_1, \dots, t_m)$ and $k_j > r_j$, $j = 1, \dots, m$.

In [5], it is given that the function $f \in S_{p,\theta}^{\bar{r}}B$ can be decomposed into Fourier series in the following form

$$\sum_{\bar{n} \in \mathbb{Z}^m, \prod_{j=1}^m n_j \neq 0} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle}.$$

Moreover, it is known (see [5]) that $\|f\|_{S_{p,\theta}^{\bar{r}}B}$ is a norm and

$$\|f\|_{S_{p,\theta}^{\bar{r}}B} \asymp \left\{ \sum_{\bar{s} \in \mathbb{Z}_+^m} 2^{\langle \bar{s}, \bar{r} \rangle \theta} \|\delta_{\bar{s}}(f)\|_p^\theta \right\}^{\frac{1}{\theta}}$$

provided $1 < p < +\infty$, $1 \leq \theta \leq +\infty$.

Therefore, in the anisotropic Lorentz space $L_{\bar{p},\bar{\theta}}^*(I^m)$, we will consider an analogous space. Suppose $S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B$ denotes the space of all functions $f \in L_{\bar{p},\bar{\theta}}^*(I^m)$ such that

$$\|f\|_{S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B} = \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} < \infty,$$

where $\bar{p} = (p_1, \dots, p_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, $\bar{\tau} = (\tau_1, \dots, \tau_m)$, $1 < p_j < \infty$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq +\infty$, $r_j > 0$, $j = 1, \dots, m$.

In this space, let's consider the unit ball (with keeping the notation)

$$S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B = \left\{ f \in L_{\bar{p},\bar{\theta}}^*(I^m) : \|f\|_{S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B} = \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \leq 1 \right\}.$$

For a fixed vector $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$, $\gamma_j > 0$, $j = 1, \dots, m$, set

$$Q_n^{\bar{\gamma}} = \cup_{\langle \bar{s}, \bar{\gamma} \rangle < n} \rho(\bar{s}), \quad T(Q_n^{\bar{\gamma}}) = \{t(\bar{x}) = \sum_{\bar{k} \in Q_n^{\bar{\gamma}}} b_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle}\},$$

$$Y^m(\bar{\gamma}, n) = \left\{ \bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \sum_{j=1}^m s_j \gamma_j \geq n \right\}.$$

Let X, Y be spaces with the norm of 2π -periodic functions of several variables. For a function $f \in X$ the following quantity is called the best M -term approximation of f [6–8]:

$$e_M(f)_X = \inf_{\bar{k}^{(j)}, b_j} \left\| f - \sum_{j=1}^M b_j e^{i\langle \bar{k}^{(j)}, \bar{x} \rangle} \right\|_X,$$

where $\{\bar{k}^{(j)}\}_{j=1}^M$ is a system of vectors $\bar{k}^{(j)} = (k_1^{(j)}, \dots, k_m^{(j)})$ with integer coordinates and b_j are arbitrary numbers.

For a given class F , let

$$e_M(F)_X = \sup_{f \in F} e_M(f)_X.$$

In the case $X = L_2$, the quantity $e_M(f)_{L_2}$ for a function of one variable was introduced by S.B. Stechkin [6] and it was used in a criteria for an absolute convergence of Fourier series by complete orthonormal systems. Order estimations of the quantity $e_M(F)_X$ were investigated by R.S. Ismagilov [7], B.Ye. Mayorov [8] (for $X = L_p$, one-dimensional case), E.S. Belinskii [9–11] (multi-dimensional case in the case $Y = L_q(I^m), X = L_p(I^m), F = W_p^r$), V.N. Temlyakov [12] (in the case $Y = L_q(I^m), F = H_p^r$), A.S. Romanyuk [13, 14], R. De Vore and R.A. DeVore [15], V.N. Temlyakov [16] (in the case $Y = L_q(I^m), F = B_{p,\theta}^r$), Dinh Dung [17]. We should note that B.S. Kashin [18] established an estimation of the quantity $e_M(f)_X$ in the case $X = L_2$ by orthonormal systems. The latest results in this direction can be found in [19–21].

In particular, the following theorem is well known.

Theorem 1 (A.S. Romanyuk [13]). Let $\bar{r} = (r_1, \dots, r_m)$, $0 < r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m$, $1 \leq p \leq 2 < q < +\infty$, and $1 \leq \theta \leq +\infty$.

1) If $r_1 > \frac{1}{p}$ then

$$e_M(S_{p,\theta}^{\bar{r}} B)_q \asymp M^{-(r_1 + \frac{1}{2} - \frac{1}{p})} (\log M)^{(\nu-1)(r_1 - \frac{1}{p} + \frac{1}{2}) + \sum_{j=2}^{\nu} (\frac{1}{2} - \frac{1}{\theta})_+};$$

2) If $\frac{1}{p} - \frac{1}{q} < r_1 < \frac{1}{p}$, then

$$e_M(S_{p,\theta}^{\bar{r}} B)_q \asymp M^{-\frac{q}{2}(r_1 + \frac{1}{q} - \frac{1}{p})} (\log M)^{(q-1)(\nu-1)(r_1 - \frac{1}{p} + \frac{q}{\theta})_+};$$

3) If $r_1 = \frac{1}{p}$, then

$$e_M(S_{p,\theta}^{\bar{r}} B)_q \asymp M^{-\frac{1}{2}} (\log^\nu M)^{\frac{1}{\theta'}},$$

where $a_+ = \max\{a, 0\}$, $\frac{1}{b} + \frac{1}{b'} = 1$.

Here $\log M$ is the logarithm with the base 2 of $M > 0$.

V.N. Temlyakov [22–24] developed a constructive method of estimation of the M -term best approximations of functions of the Nikol'skii-Besov's class $S_{p,\tau}^{\bar{r}} B$ in the space $L_q(I^m)$ in the case $1 < p < q < \infty$. This method is based on greedy algorithms.

The main goal of the present paper is to find the exact order of the best M -term approximation of a function in the class $S_{p,\theta,\bar{\tau}}^{\bar{r}} B$ in the Lorentz spaces with anisotropic norm in the case $1 < p_j < q_j \leq 2$, $j = 1, \dots, m$, and in the case $1 < p_j < 2 \leq q_j < \infty$, $j = 1, \dots, m$, to give a constructive proof for an upper bound for the quantity

$$e_M(S_{p,\theta,\bar{\tau}}^{\bar{r}} B)_{q,\theta}.$$

Let us denote by $C(p, q, r, y)$ positive quantities which depend on the parameters in the parentheses, which are, in general, are distinct in distinct formulas. $A(y) \asymp B(y)$ means that there are positive C_1, C_2 such that $C_1 \cdot A(y) \leq B(y) \leq C_2 \cdot A(y)$.

S.1. Auxiliary results

To prove the main results, we need the following auxiliary results.

Lemma 1 [25]. Suppose $\alpha \in (0, +\infty)$ and $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$, $\bar{\gamma}' = (\gamma'_1, \dots, \gamma'_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, $\theta_j \in [1, +\infty)$, $j = 1, \dots, m$, $1 = \gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \leq \dots \leq \gamma_m$, $1 = \gamma'_j = \gamma_j$, $j = 1, \dots, \nu$, and $1 = \gamma'_j < \gamma_j$, $j = \nu + 1, \dots, m$. Then the following relation holds

$$\left\| \left\{ 2^{-\alpha \langle \bar{s}, \bar{\gamma} \rangle} \right\}_{\bar{s} \in Y^m(\bar{\gamma}', n)} \right\|_{l_{\bar{\theta}}} \asymp 2^{-n\alpha} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

Remark. For the case $\theta_1 = \dots = \theta_m$, Lemma 1 was proved by V.N. Temlyakov [12]. In what follows we denote by $\chi_{\mathcal{X}(n)}(\bar{s})$ the characteristic function of the set $\mathcal{X}(n) = \{\bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \langle \bar{s}, \bar{\gamma} \rangle = n\}$.

Lemma 2 ([25], Lemma 2.3 and [26], Lemma 4). Let $\bar{\tau} = (\tau_1, \dots, \tau_m)$, $1 \leq \tau_j < +\infty$, and $j = 1, \dots, m$. Then the following relation holds

$$\left\| \left\{ \chi_{\mathcal{X}(n)}(\bar{s}) \right\}_{\bar{s} \in \mathcal{X}(n)} \right\|_{l_{\bar{\tau}}} \asymp n^{\sum_{j=2}^m \frac{1}{\tau_j}}, \quad n \in \mathbb{N}.$$

Theorem 2 [27]. Let $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, $\bar{r} = (r_1, \dots, r_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, $\bar{\tau} = (\tau_1, \dots, \tau_m)$, $1 \leq p_j < 2 < q_j$, $j = 1, \dots, m$, $1 \leq \theta_j, \tau_j < +\infty$, $0 < r_1 + \frac{1}{q_1} - \frac{1}{p_1} = \dots = r_\nu + \frac{1}{q_\nu} - \frac{1}{p_\nu} < r_{\nu+1} + \frac{1}{q_{\nu+1}} - \frac{1}{p_{\nu+1}} \leq \dots \leq r_m + \frac{1}{q_m} - \frac{1}{p_m}$.

1) If $r_j > \frac{1}{p_j}$, $j = 1, \dots, m$, $(r_1 - \frac{1}{p_1})\frac{1}{q_1} < (r_j - \frac{1}{p_j})\frac{1}{q_1}$, $j = \nu + 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}} \asymp M^{-\left(r_1 + \frac{1}{2} - \frac{1}{p_1}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{2}\right) + \sum_{j=2}^{\nu} \left(\frac{1}{2} - \frac{1}{\tau_j}\right)};$$

2) If $\frac{1}{p_j} - \frac{1}{q_j} < r_j < \frac{1}{p_j}$, $\theta_j < \tau_j$, $j = 1, \dots, m$, $(r_1 - \frac{1}{p_1})\frac{1}{q_1} < (r_j - \frac{1}{p_j})\frac{1}{q_1}$, $j = \nu + 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}} \asymp M^{-\frac{q_1}{2} \left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{q_1 \left(r_1 - \frac{1}{p_1}\right) \sum_{j=2}^{\nu} \frac{1}{\theta_j} + \sum_{j=2}^{\nu} \frac{1}{\tau_j}};$$

3) If $\nu \leq \mu$, $r_j = \frac{1}{p_j}$, $j = 1, \dots, \mu$ and $r_j > \frac{1}{p_j}$, $j = \mu + 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}} \asymp M^{-\frac{1}{2}} (\log M)^{\sum_{j=1}^{\mu} \frac{1}{\tau_j}}.$$

Theorem 3 [25]. Let $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, $\bar{\theta}^{(1)} = (\theta_1^{(1)}, \dots, \theta_m^{(1)})$, $\bar{\theta}^{(2)} = (\theta_1^{(2)}, \dots, \theta_m^{(2)})$ and $1 \leq p_j < q_j < +\infty$, $1 \leq \theta_j^{(1)}, \theta_j^{(2)} < +\infty$, $j = 1, \dots, m$. If $f \in L_{\bar{p}, \bar{\theta}^{(1)}}^*(I^m)$ and the quantity

$$\sigma(f) \equiv \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m^{(2)} \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \left[\dots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1^{(2)} \left(\frac{1}{p_1} - \frac{1}{q_1}\right)} \left(\|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta_1^{(2)}} \right]^{\theta_1^{(2)}} \dots \right]^{\frac{\theta_m^{(2)}}{\theta_{m-1}^{(2)}}} \right\}_{\theta_m^{(2)}}^{\frac{1}{\theta_m^{(2)}}}$$

is finite, then $f \in L_{\bar{q}, \bar{\theta}^{(2)}}^*(I^m)$ and

$$\|f\|_{\bar{q}, \bar{\theta}^{(2)}}^* \leq C(p, q, \theta) \cdot \sigma(f).$$

S. 2. Estimates of the best M -term approximations of functions

Theorem 4. Let $1 < q_j < 2 \leq p_j < +\infty$, $1 < \theta_j, \lambda_j < +\infty$, $j = 1, \dots, m$. If $f \in L_{\bar{q}, \bar{\theta}}^*(I^m)$, then

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C(q, \theta, p, m) \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \left[\dots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 \left(\frac{1}{p_1} - \frac{1}{q_1}\right)} \left(\|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\lambda}}^* \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}_{\theta_m}^{\frac{1}{\theta_m}}.$$

Proof. Firstly, we will prove the theorem for the case $p_j = \lambda_j^{(1)} = 2$, $j = 1, \dots, m$. Then

$$\|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\lambda}}^* = \|\delta_{\bar{s}}(f)\|_2, \quad \bar{s} \in \mathbb{Z}_+^m.$$

By the conditions of the theorem $f \in \dot{L}_{\bar{q}, \bar{\theta}}^*(I^m)$, $1 < q_j, \theta_j < +\infty$, $j = 1, \dots, m$. Therefore [1]

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq \sup_{\|g\|_{\bar{q}', \bar{\theta}'}^* \leq 1} \int_{I^m} f(\bar{x})g(\bar{x})d\bar{x}, \quad (1)$$

where $\bar{q}' = (q'_1, \dots, q'_m)$, $\bar{\theta}' = (\theta'_1, \dots, \theta'_m)$, $\frac{1}{q_j} + \frac{1}{q'_j} = 1$, $\frac{1}{\theta_j} + \frac{1}{\theta'_j} = 1$, $j = 1, \dots, m$.

From the inequality (1), we have

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq \sup_{\|g\|_{\bar{q}', \bar{\theta}'}^* \leq 1} \sum_{\bar{s} \in \mathbb{Z}_+^m} \int_{I^m} \delta_{\bar{s}}(f, \bar{x})\delta_{\bar{s}}(g, \bar{x})d\bar{x}. \quad (2)$$

Since by the assumption of the theorem $1 < q_j < 2$, $j = 1, \dots, m$, we have $2 < q'_j < \infty$, $j = 1, \dots, m$. Therefore, by Theorem 3 for $p_j = \theta_j^{(1)} = 2$, $j = 1, \dots, m$, we obtain

$$\|g\|_{\bar{q}', \bar{\theta}'}^* \leq C_0 \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q'_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}'}}. \quad (3)$$

Let's introduce the following notation

$$U_{\bar{q}', \bar{\theta}'} = \left\{ g \in L_{\bar{q}', \bar{\theta}'}^*(I^m) : C_0 \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q'_j})} \|\delta_{\bar{s}}(g)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}'}} \leq 1 \right\}.$$

Then it follows from the inequality (3) that the set $U_{\bar{q}', \bar{\theta}'}$ is a subset of the unit ball of the space $L_{\bar{q}', \bar{\theta}'}^*(I^m)$. Therefore, the formula (2) implies that

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \sup_{g \in U_{\bar{q}', \bar{\theta}'}} \sum_{\bar{s} \in \mathbb{Z}_+^m} \int_{I^m} \delta_{\bar{s}}(f, \bar{x})\delta_{\bar{s}}(g, \bar{x})d\bar{x}. \quad (4)$$

If $g \in L_{\bar{q}', \bar{\theta}'}^*(I^m)$, such that $\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}$, $\bar{s} \in \mathbb{Z}_+^m$ and

$$\left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q'_j})} b_{\bar{s}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}'}} \leq 1,$$

then from inequality (4) we obtain

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \sup_{\{b_{\bar{s}}\}} \sum_{\bar{s} \in \mathbb{Z}_+^m} \sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x})\delta_{\bar{s}}(g, \bar{x})d\bar{x}. \quad (5)$$

Now let's prove that the following inequality holds

$$\sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x})\delta_{\bar{s}}(g, \bar{x})d\bar{x} \geq \|\delta_{\bar{s}}(f)\|_2 b_{\bar{s}} \quad (6)$$

for each $\bar{s} \in \mathbb{Z}_+^m$. So consider the function

$$g_0(\bar{x}) = \sum_{\bar{k} \in \rho(\bar{s})} a_{\bar{k}}(g_0) e^{i\langle \bar{k}, \bar{x} \rangle},$$

where

$$a_{\bar{k}}(g_0) = \|\delta_{\bar{s}}(f)\|_2^{-1} a_{\bar{k}}(f) b_{\bar{s}}.$$

Then $\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}$, $\bar{s} \in \mathbb{Z}_+^m$. Hence, by the Parseval's equality, we have

$$\begin{aligned} \sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x} &\geq \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g_0, \bar{x}) d\bar{x} = \\ &= \|\delta_{\bar{s}}(f)\|_2^{-1} b_{\bar{s}} \sum_{\bar{k} \in \rho(\bar{s})} a_{\bar{k}}^2(f) = \|\delta_{\bar{s}}(f)\|_2 b_{\bar{s}}, \end{aligned}$$

which proves the formula (6).

Next, it follows from formulas (5) and (6) that

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \sup_{\{b_{\bar{s}}\}} \sum_{\bar{s} \in \mathbb{Z}_+^m} b_{\bar{s}} \|\delta_{\bar{s}}(f)\|_2. \quad (7)$$

Suppose

$$\begin{aligned} \sigma_{\bar{s}_{m-j}}(f)_{\bar{\theta}_j} &= \\ &= \left\{ \sum_{s_j=1}^{\infty} 2^{s_j \theta_j (\frac{1}{2} - \frac{1}{q_j})} \left[\dots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 (\frac{1}{2} - \frac{1}{q_1})} \|\delta_{\bar{s}}(f)\|_2^{\frac{\theta_1}{\theta_1}} \dots \right]^{\frac{\theta_2}{\theta_2}} \dots \right]^{\frac{\theta_j}{\theta_j-1}} \right\}^{\frac{1}{\theta_j}}, \end{aligned}$$

where $\bar{s}_{m-j} = (s_{j+1}, \dots, s_m)$, $\bar{\theta}_j = (\theta_1, \dots, \theta_j)$, $j = 1, \dots, m-1$.

Consider the following sequence

$$\begin{aligned} b_{\bar{s}} &= \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}^{\frac{\theta_m}{\theta_m}} \prod_{j=1}^m (\sigma_{\bar{s}_{m-j}}(f)_{\bar{\theta}_j})^{\theta_{j+1} - \theta_j} \times \\ &\quad \times \|\delta_{\bar{s}}(f)\|_2^{\theta_1 - 1} \prod_{j=1}^m 2^{s_j (\frac{1}{2} - \frac{1}{q_j}) \theta_j} \end{aligned}$$

for $\bar{s} \in \mathbb{Z}_+^m$. Then

$$\left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{2} - \frac{1}{q_j})} b_{\bar{s}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}} = 1$$

and

$$\sum_{\bar{s} \in \mathbb{Z}_+^m} b_{\bar{s}} \|\delta_{\bar{s}}(f)\|_2 = \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}.$$

Therefore, from the inequality (7), we get

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}. \quad (8)$$

Next, if $2 < p_j < \infty$, $j = 1, \dots, m$, then, by applying the inequality of different metrics for trigonometric polynomials [28, 29], we obtain from the estimate (8) the following

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j}\right)} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\lambda}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}$$

for $1 < q_j < 2 \leq p_j < \infty$, $1 < \theta_j, \lambda_j < \infty$, $j = 1, \dots, m$. The theorem has been proved.

Suppose $y_+ = \max\{0, y\}$.

Theorem 5. Let $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, $\bar{r} = (r_1, \dots, r_m)$, $1 \leq p_j < q_j \leq 2$, $j = 1, \dots, m$, $1 \leq \theta_j^{(1)}, \theta_j^{(2)} < +\infty$, $1 \leq \tau_j \leq +\infty$, $0 < r_1 + \frac{1}{q_1} - \frac{1}{p_1} = \dots = r_\nu + \frac{1}{q_\nu} - \frac{1}{p_\nu} < r_{\nu+1} + \frac{1}{q_{\nu+1}} - \frac{1}{p_{\nu+1}} = \dots = r_m + \frac{1}{q_m} - \frac{1}{p_m}$.

1. If $\theta_j^{(2)} \leq \tau_j, j = 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{r}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}^{(2)}} \asymp M^{-\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{q_1}\right) + \sum_{j=2}^{\nu} \left(\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j}\right)}.$$

2. If $\theta_1^{(2)} = \dots = \theta_m^{(2)} = \theta > \tau = \tau_1 = \dots = \tau_m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{r}}^{\bar{r}} B \right)_{\bar{q}, \theta} \leq M^{-\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log^{\nu-1} M)^{\left(r_1 - \frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{\theta} - \frac{1}{\tau}\right)_+}.$$

3. If $\tau_j \leq \theta_j^{(2)}, j = 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{r}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}^{(2)}} \geq C M^{-\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{q_1}\right) + \sum_{j=2}^{\nu} \left(\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j}\right)}.$$

Proof. The upper bound estimation of the quantity $e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{r}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}^{(2)}}$ was proved in [30]. Let us consider the lower bound.

For a number $M \in \mathbb{N}$ choose a natural number n such that $M \asymp 2^n n^{m-1}$ and $2^n n^{m-1} \geq 4M$.

Consider the function

$$f_0(\bar{x}) = n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \sum_{\langle \bar{s}, \bar{\gamma} \rangle = n} \prod_{j=1}^m 2^{-s_j \left(r_j + 1 - \frac{1}{p_j}\right)} \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Then, by Lemma 2,

$$\begin{aligned} & \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f_0)\|_{\bar{p}, \bar{\lambda}}^* \right\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{r}}} = \\ & = n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \prod_{j=1}^m 2^{-s_j \left(r_j + 1 - \frac{1}{p_j}\right)} \left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\lambda}}^* \right\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{r}}} \leq \\ & \leq C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \prod_{j=1}^m 2^{-s_j \left(r_j + 1 - \frac{1}{p_j}\right)} \prod_{j=1}^m 2^{s_j \left(1 - \frac{1}{p_j}\right)} \right\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{r}}} = \\ & = C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \{1\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{r}}} \leq C_0. \end{aligned}$$

Thus, the function $f_0 \in S_{\bar{p}, \bar{\theta}^{(1)}, \bar{r}}^{\bar{r}} B$.

Let Ω_M be an arbitrary collection in M of m -dimensional vectors $\{\bar{k}^{(1)}, \dots, \bar{k}^{(M)}\}$ with integer coordinates. For every vector \bar{s} , that satisfies the condition $\langle \bar{s}, \bar{\gamma} \rangle = n$, we consider the set $\Omega_M \cap \rho(\bar{s})$. Then, according to the choice of the number n , the set S of vectors \bar{s} such that $\langle \bar{s}, \bar{\gamma} \rangle = n$ and $|\Omega_M \cap \rho(\bar{s})| \leq \frac{1}{2} |\rho(\bar{s})|$ contains at least half of all vectors \bar{s} such that $\langle \bar{s}, \bar{\gamma} \rangle = n$. Hence $|S| \asymp n^{m-1}$.

Let $T(\bar{x})$ be an arbitrary polynomial with a collection of harmonics from Ω_M . Then, by Theorem 4, for $p_j = \lambda_j = 2$, $j = 1, \dots, m$, we get

$$\begin{aligned} \|f_0 - T\|_{\bar{q}, \bar{\theta}(2)}^* &\geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f_0 - T)\|_2 \right\}_{\langle \bar{s}, \bar{\gamma} \rangle = n} \right\|_{l_{\bar{\theta}(2)}} \geq \\ &\geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f_0 - T)\|_2 \right\}_{\bar{s} \in S} \right\|_{l_{\bar{\theta}(2)}} \geq \\ &\geq C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \prod_{j=1}^m 2^{\frac{s_j}{2}} \right\}_{\bar{s} \in S} \right\|_{l_{\bar{\theta}(2)}} \geq \\ &\geq C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1}) \langle \bar{s}, \bar{\gamma} \rangle} \right\}_{\bar{s} \in S} \right\|_{l_{\bar{\theta}(2)}} = C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}(2)}}. \end{aligned}$$

Applying the Holder's inequality ($\frac{1}{\theta_j^{(2)}} + \frac{1}{\theta_j^{(2)'}} = 1$, $j = 1, \dots, m$) and Lemma 4, we have

$$\begin{aligned} |S| &= \sum_{\bar{s} \in S} 1 \leq \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}(2)}} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}(2)'}} \leq \\ &\leq \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}(2)}} \|\{1\}_{\langle \bar{s}, \bar{\gamma} \rangle = n}\|_{l_{\bar{\theta}(2)'}} \leq C n^{\sum_{j=2}^m \frac{1}{\theta_j^{(2)'}}} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}(2)}}. \end{aligned}$$

Since $|S| \asymp n^{m-1}$, we obtain

$$n^{\nu-1} \leq C n^{\sum_{j=2}^m \frac{1}{\theta_j^{(2)'}}} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}(2)}}.$$

Hence

$$n^{\sum_{j=2}^m \frac{1}{\theta_j^{(2)'}}} \leq C \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}(2)}}.$$

Therefore,

$$\|f_0 - T\|_{\bar{q}, \bar{\theta}(2)}^* \geq C 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} n^{\sum_{j=2}^m (\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})}$$

for any polynomial $T(\bar{x})$ with a collection of harmonics from Ω_M . Hence

$$\begin{aligned} e_M \left(S_{\bar{p}, \bar{\theta}(1), \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}(2)} &\geq C 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} n^{\sum_{j=2}^m (\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})} \geq \\ &\geq C M^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} (\log M)^{(m-1)(r_1 - \frac{1}{p_1} + \frac{1}{q_1}) + \sum_{j=2}^m \left(\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j} \right)}. \end{aligned} \quad (9)$$

Since $\nu \leq m$, then we obtain the lower bound estimation from (9). The theorem has been proved.

V.N. Temlyakov [22, 23] developed a constructive method of estimation of the best M -term approximations functions from Nikol'skii-Besov's classes $S_{p, \tau}^{\bar{r}} B$ in the space $L_q(I^m)$ in the case $1 < p < 2 \leq q < \infty$. Let us recall some notations.

Let $\bar{r} = (r_1, \dots, r_m)$, $r_j > 0$, $j = 1, \dots, m$, and $l \in \mathbb{N} \cap \{0\} = \mathbb{N}_0$. Let $r = r_1$ and for a function $f \in L_1(I^m)$ suppose

$$\begin{aligned} f_{l, \bar{r}}(\bar{x}) &= \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \delta_{\bar{s}}(f, \bar{x}), \quad \bar{x} \in I^m; \\ \|f_{l, \bar{r}}\|_A &= \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \sum_{\bar{k} \in \rho(\bar{s})} |a_{\bar{k}}(f)|. \end{aligned}$$

V.N. Temlyakov [22] considered the class

$$W_A^{\bar{r}, a, b} = \left\{ f \in L_1(I^m) : \|f_{l, \bar{r}}\|_A \leq 2^{-la} l^{(\nu-1)b} \right\},$$

where $a > 0, b > 0$.

Lemma 3 [22]. Let $2 \leq p < +\infty$ and $a > 0, b > 0$. Then there exists a constructive method based on the greedy algorithms which leads to the following estimation

$$e_M(W_A^{\bar{r}, a, b})_p \leq CM^{-a-\frac{1}{2}} (\log M)^{(\nu-1)(a+b)}.$$

Theorem 6. Let $1 < p_j < 2 \leq q < +\infty, 1 < \theta_j < \infty, 1 \leq \tau_j \leq \infty, j = 1, \dots, m, 0 < r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m, r_1 > \frac{1}{p_1}$ and $\frac{r_1}{p_j} = \frac{r_j}{p_1}, j = 1, \dots, \nu$ and $\frac{r_1}{p_j} < \frac{r_j}{p_1}, j = \nu + 1, \dots, m$. Then the following relation holds

$$e_n(S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B)_q \asymp n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r-\frac{1}{p_1}) + \sum_{j=2}^{\nu} \frac{1}{\tau_j}},$$

where $\tau_j' = \frac{\tau_j}{\tau_j-1}, j = 1, \dots, m$.

Proof. Let $f \in S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B$. By the condition of the theorem $1 < p_j < 2, j = 1, \dots, m$, then there exists a number p_0 such that $1 < p_j < p_0 < 2, j = 1, \dots, m$. Therefore, by applying the Holder's inequality ($\frac{1}{p_0} + \frac{1}{p_0} = 1$) and the Hausdorff-Young's Theorem [31; 211], we get

$$\begin{aligned} \|f_{l, \bar{r}}\|_A &\leq \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \prod_{j=1}^m 2^{(s_j-1)\frac{1}{p_0}} \left(\sum_{\bar{k} \in \rho(\bar{s})} |a_{\bar{k}}(f)|^{p_0} \right)^{\frac{1}{p_0}} \leq \\ &\leq C \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \prod_{j=1}^m 2^{s_j \frac{1}{p_0}} \|\delta_{\bar{s}}(f)\|_{p_0}. \end{aligned} \tag{10}$$

Now, since $p_j < p_0, j = 1, \dots, m$, then by applying the inequality of different metrics for trigonometric polynomials (see [28, 29]) and the Holder's inequality, we obtain from (10) the following

$$\begin{aligned} \|f_{l, \bar{r}}\|_A &\leq C \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \prod_{j=1}^m 2^{s_j \frac{1}{p_j}} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}}^* \leq \\ &\leq C \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}}^* \right\}_{r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \right\|_{l_{\bar{r}}} \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{p_j} - r_j)} \right\}_{r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \right\|_{l_{\bar{r}}}. \end{aligned} \tag{11}$$

Suppose $\gamma_j = \frac{r_j}{r_1}, \gamma_j' = \frac{r_j - \frac{1}{p_j}}{r_1 - \frac{1}{p_1}}, j = 1, \dots, m$. Then, by the assumption of the theorem, $1 = \gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \leq \dots \leq \gamma_m$ and $\gamma_j' = \gamma_j, j = 1, \dots, \nu, \gamma_j = \gamma_j', j = \nu + 1, \dots, m$. Then using Lemma 1 we have

$$\begin{aligned} &\left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{p_j} - r_j)} \right\}_{r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \right\|_{l_{\bar{r}}} \leq \\ &\leq \left\| \left\{ 2^{-(r_1 - \frac{1}{p_1}) \langle \bar{s}, \bar{\gamma}' \rangle} \right\}_{\langle \bar{s}, \bar{\gamma}' \rangle \geq l} \right\|_{l_{\bar{r}}} \leq C 2^{-l(r_1 - \frac{1}{p_1})} l^{\sum_{j=2}^{\nu} \frac{1}{\tau_j}}, \end{aligned} \tag{12}$$

where $\frac{1}{\tau_j} = 1 - \frac{1}{\tau_j}, j = 1, \dots, m$.

Next, it follows from the inequalities (11) and (12) that

$$\|f_{l, \bar{r}}\|_A \leq C 2^{-l(r_1 - \frac{1}{p_1})} l^{\sum_{j=2}^{\nu} \frac{1}{\tau_j}}$$

for any function $f \in S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B$ in the case $r_1 > \frac{1}{p_1}$. Thus, the function $f \in S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B$ belongs to the class $W_A^{\bar{r}, a, b}$ in the case $a = r_1 - \frac{1}{p_1}$ and $b = \frac{1}{\nu-1} \sum_{j=2}^{\nu} \frac{1}{\tau_j}$. Hence, by Lemma 3, we have

$$e_n(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B)_q \leq C e_n(W_A^{\bar{r}, a, b})_q \leq C n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r-\frac{1}{p_1})+\sum_{j=2}^{\nu} \frac{1}{\tau_j}}$$

in the case $1 < p_j < 2 \leq q < +\infty$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq \infty$, $j = 1, \dots, m$, $r_1 > \frac{1}{p_1}$. It proves the upper bound estimation.

Now let's consider the lower bound estimation. Since $2 \leq q < \infty$, then $\|f\|_2 \leq C \|f\|_q$, $f \in L_q(I^m)$. Therefore

$$e_n(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B)_q \geq C e_n(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B)_2.$$

Now, by letting $\theta_j = q = 2$, $j = 1, \dots, m$, in Theorem 5, we get

$$e_n(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B)_2 \geq C n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r+\frac{1}{2}-\frac{1}{p_1})+\sum_{j=2}^{\nu} (\frac{1}{2}-\frac{1}{\tau_j})}.$$

Hence

$$\begin{aligned} e_n(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B)_q &\geq C n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r+\frac{1}{2}-\frac{1}{p_1})+\sum_{j=2}^{\nu} (\frac{1}{2}-\frac{1}{\tau_j})} \geq \\ &\geq C n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r-\frac{1}{p_1})+\sum_{j=2}^{\nu} \frac{1}{\tau_j}}. \end{aligned}$$

It proves the theorem.

Remark. Note that the upper bound estimation in the case $p_j = \theta_j = p$, $j = 1, \dots, m$, in Theorem 6 was proved by V.N. Temlyakov [22] using the constructive method.

In [32] obtained the exact estimation of the best M -term approximations of Nikol'skii's and Besov's classes in the Lebesgue space with the mixed norm.

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Г. Ақышев

Лоренц кеңістігінде функциялардың ең жақсы M -мүшелі жуықтауларын конструктивті әдістермен бағалау

Мақалада көпайнымалы периодты функциялардың аралас нормалы Лоренц кеңістігі, Никольский-Бесовтың функционалдық класы және функцияның ең жақсы M -мүшелі жуықтауы қарастырылды. Сондай-ақ бір Лоренц кеңістігіндегі функцияның басқасында жатуының жеткілікті шарты тағайындалған. Никольский-Бесов класындағы функциялардың ең жақсы M -мүшелі жуықтауларының жоғарыдан және төменнен бағалаулары алынған. Жоғарыдан бағалауды дәлелдеу үшін В.Н. Темляков құрған жаңа конструктивтік әдіс қолданылды.

Кілт сөздер: Лоренц, Никольский-Бесов класы, M -терминдердің ең жақсы жақтары, жуықтаулар, жеткілікті шарттар, бағалау.

Г. Акишев

Оценки наилучших M -членных приближений функций в пространстве Лоренца конструктивными методами

В статье рассмотрены пространство Лоренца периодических функций многих переменных с анизотропной нормой; функциональный класс Никольского-Бесова и наилучшее M -членное приближение функции. Установлены достаточные условия принадлежности функции из одного пространства Лоренца в другое. Получены оценки сверху и снизу наилучших M -членных приближений функций из класса Никольского-Бесова. Для доказательства оценки сверху использован новый конструктивный метод, разработанный В.Н. Темляковым.

Ключевые слова: Лоренц, класс Никольского-Бесова, лучшие аппроксимации M -термов, приближение, достаточные условия, оценка.

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