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The structure of normal subsets of polyhedral cone

The structure of normal convex subsets of polyhedral cone K in normalized space is in investigated. The normality of the subset $\Omega \subset K$ (in the sense of the cone K) is determined by the condition $\overline{\Omega - K} \cap K = \Omega$ (a line over a set means taking a topological closure). The conical shell of finite number of rays mean the polyhedrons of the cone, which are extreme rays. The structure of normal sets were studied from the geometric point of view. It is shown that every normal subset Ω of a polyhedral cone can be divided into a sum of two subsets, one of which is a bounded normal subset (in the sense of some subcone in K) and the second – the subcone K contained in the set Ω (it is unbounded, if Ω is unbounded).

Keywords: cone, ray, normal, conical shell, polyhedral cone, forming rays, closed set, normal subset of cone, null cone.

Let $(X, \| \quad \|)$ is the real normalized space.

For the subset of $G \subset X$ intoduce the notation: G° — interior of G, $\bar{G}$ — closure of G, frG — frontier of G. Everywhere $\varnothing$ - empty set.

All the operation will be entered in the space X:

$$G_1 + G_2 = \{x = x_1 + x_2, x_1 \in G_1, x_2 \in G_2\}; \quad G_1, G_2 \subset X;$$

$$\alpha G = \{x = \alpha x, x \in G\}, \quad G \subset X, \quad \alpha - \text{real}.$$

The subset $G \subset X$ is called convex, if $x = \alpha x_1 + (1-\alpha)x_2 \in G$ for any $x_1, x_2 \in G$ и $0 \leqslant \alpha \leqslant 1$. The subset $K \subset X$ is called cone (convex), if the task is done:

$$\alpha x \in K \quad \forall x \in K, \quad \alpha \geqslant 0;$$

$$x_1 + x_2 \in K \quad \forall x_1, x_2 \in K.$$

It is obvious, that any cone 0 has own point x and outgoing ray $L = \{\alpha x, \alpha \geqslant 0\}$ from 0. The cone K is called bodily, if $K^\circ \neq \varnothing$; salient, if $K \cap (-K) = \{0\}$. The conical shell coG of set $G \subset X$ is called set of elements

$$coG = \{x = \sum_{i=1}^{n} \alpha_i x_i, \quad n - \text{any natural}; \quad x_i \in G \quad \text{и} \quad \alpha_i \geqslant 0 \quad \forall i = 1, \ldots, n\}.$$

It is obvious that the conical shell is cone. We say it will pull on the set G.

If $G = \varnothing$, then we consider that $K = coG = \{0\}$.

The cone K ([1]) is called polyhedral, if it is cone shell of finite number of outgoing rays from 0. According to the [2] we can see, that every polyhedral cone is closure and bodily in its linear shell. The ray L of cone K is called extreme ray of cone, if from equality $x = x_1 + x_2$, where $x \in L$, $x_1, x_2 \in K$, follows, that $x_1, x_2 \in L$. According to the [1] we can see, that the minimal (by inclusion) set of rays, it's conical shell is cone K, forming the extreme rays.

The extreme rays of cone we will call forming rays. If $\{L_i\}_{i \in I}$ is forming rays of polyhedral cone K, $\{e_i\}_{i \in I}$ is units (i.e. $\|e_i\| = 1$) directional vectors of this rays (I — finite set of indices), then

$$K = \{x = \sum_{i \in I} \alpha_i e_i, \quad \alpha_i \geqslant 0 \quad \forall i \in I\}.$$

Let K is polyhedral cone, $\{L_i\}_{i \in I}$ is forming rays of cone K. A sub subcone K_1 of cone K, we will be understand that any polyhedral cone, which is formed by rays $\{L_i\}_{i \in I_1 \subset I}$. If $I_1 = \varnothing$, then the subcone, which determined by set of indices I_1, suppose equal to $\{0\}$. If we speak about o cone in this work we will view a polyhedral cones. Let K is cone, $\{L_i\}_{i\in I}$ is its forming rays. For set $\Omega \subset K$ we think that

$$\beta_i = sup\{\beta \geqslant 0 : \beta e_i \in \Omega\}, \quad i \in I.$$

If $\Omega \cap L_i = \varnothing$, then we think that $\beta_i = 0$. We say that the set $\Omega \subset K$ has a (0)- property with respect to subcone spreat on rays $\{L_i\}_{i\in I_1 \subset I}$, if $0 < \beta_i < \infty$. The set $\Omega \subset K$, Ω is called normal (that means cone K) set, if Ω is convex and rightly the equal $\overline{\Omega - K} \cap K = \Omega$.

Note 1. Directly from the definition following, that any normal subset be closure.

Note 2. If Ω is normal subset of cone, $x \in \Omega$, the segment will

$$[0, x] = \{\alpha x : 0 \le \alpha \le 1\} \subset \Omega.$$

Its objectively that $x \in \Omega \subset K$, wherefrom $\alpha x \in K$. In other case, $\alpha x = x - (1-\alpha)x$, where $x \in \Omega$, $(1-\alpha)x \in K$, i.e. $\alpha x \in \Omega - K$, according this

$$\alpha x \in (\Omega - K) \cap K \subset \overline{\Omega - K} \cap K = \Omega.$$

Consider the question about structure of normal subsets of polyhedral cone.

Rightly Proposition. Let Ω is normal (not be cone) subset of cone $K \Rightarrow \Omega = G + K_1$, where K_1 is greatest by inclusion contained in Ω подконус K; $G = \Omega \cap K_2$ (where K_2 is subset in K) and G is finite normal set and it has a (0)-property with respect to cone K_2.

Proof of proposition. Let K is polyhedral cone, $\{L_i\}_{i\in I}$ is set its forming rays.

1. We will consider the case of finite set Ω, i.e. $\exists C > 0 : \|x\| \le C \quad \forall x \in \Omega$.

If $\Omega = \{0\}$, then Ω is null cone, this case in proposition excluded.

Let $\Omega \ne \{0\}$, then $\forall x \in \Omega$ $(x \ne 0)$ we have

$$x = \sum_{i\in I} \alpha_i e_i \quad \text{end} \quad \exists \alpha_{i_0} > 0 \quad \text{other wise} \quad x = 0.$$

Rightly the note $\alpha_{i_0} e_{i_0} = x - \sum\limits_{i \ne i_0} \alpha_i e_i$, where from

$$\alpha_{i_0} e_0 \in (x - K) \cap K \subset (\Omega - K) \cap K \subset \overline{\Omega - K} \cap K = \Omega$$

by the normality Ω with respect to K.

As $\alpha_{i_0} e_0 \in \Omega$, a $\beta_{i_0} = sup\{\beta \geqslant 0 : \beta e_i \in \Omega\}$, то $\beta_{i_0} \ge \alpha_{i_0} > 0$ and that's the

$$I_2 = \{i \in I : \beta_i > 0\} \ne \varnothing.$$

Let K_2 is cone with forming rays $\{L_i\}_{i\in I_2}$. Following from definition, that Ω has a (0)-property with respect to cone K_2. We will show, that $\Omega \subset K_2$. If suppose, that $\Omega \not\subset K_2$, то $\exists x \in \Omega : x \notin K_2$. From inclusion $x \in \Omega \subset K$ we have

$$x = \sum_{i\in I_2} \alpha_i e_i + \sum_{i \in I \setminus I_2} \alpha_i e_i.$$

Since $x \notin K_2$, then $\exists i_0 \in I \setminus I_2 : \alpha_{i_0} > 0$, tnen

$$\alpha_{i_0} e_{i_0} = x - \sum_{i \ne i_0} \alpha_i e_i \in (x - K) \cap K \subset \overline{\Omega - K} \cap K = \Omega,$$

because $\beta_{i_0} \ge \alpha_{i_0} > 0$, where from $i_0 \in I_2$, that this is contrary to inclusion $i_0 \in I \setminus I_2$, according this $\Omega \subset K_2$ and $G = \Omega \cap K_2 = \Omega$.

Since Ω is finite set, we get that

$$\beta_i = sup\{\beta \geqslant 0 : \beta e_i \in \Omega\} \le c < \infty.$$

According this $0 < \beta_i < \infty$ for $\forall i \in I_2$, where from following, that $G = \Omega$ has a (0)-property, with respect to cone K_2.

We have the inclusion $G \subset \overline{G-K_2} \cap K_2$ from ratio $G = \Omega \subset K_2$. Check the inverse inclusion. From condition of normality Ω with respect to cone K, i.e. the equal $\overline{\Omega - K} \cap K = \Omega$ and the inclusion $K_2 \subset K$, we have:

$$\overline{G-K_2} \cap K_2 = \overline{\Omega - K_2} \cap K_2 \subset \overline{\Omega - K} \cap K = \Omega = G.$$

As contained in Ω the cone we take $K_1 = \{0\}$.

2. Now let the set Ω is undefinite. We show, that Ω contains a subcone.

By definition $\beta_i = sup\{\beta \geqslant 0 : \beta e_i \in \Omega\}, \quad \forall i \in I$, so $\exists i \in I : \beta_i = \infty$.

Really, if it is not, i.e. $\beta_i < \infty, \quad \forall i \in I$, then

$$x = \sum_{i\in I} \alpha_i e_i \quad \text{and} \quad \|x\| \le \sum_{i\in I} \alpha_i \le \sum_{i \in I} \beta_i, \quad \text{где} \sum_{i\in I} \beta_i = \beta < \infty,$$

i.e. $\|x\| \le \beta \quad \forall x \in \Omega$, where we get a contradiction of undefinite of set Ω.

Let $I_1 = \{i \in I : \beta_i = \infty\}$, we showed, that $I_1 \neq \varnothing$ and let K_1 is subcone with forming rays $\{L_i\}_{i\in I_1}$. We wii show, that $K_1 \subset \Omega$.

This follows from the fact, that $\beta_i = sup\{\beta \geqslant 0 : \beta e_i \in \Omega\} = \infty$ for $\forall i \in I_1$, i.e. if $i \in I_1$ и $x = \alpha e_i \in L_i$, that $\exists \beta > \alpha : \beta e_i \in \Omega$, we have a ratio from note 2: $[0, \beta e_i] \subset \Omega$.

By condition $0 \le \alpha < \beta$, so $\alpha x \in [0, \beta e_i] \subset \Omega \quad \forall \alpha \ge 0$, where from the ray $L_i = \{\alpha e_i, \alpha \ge 0\} \subset \Omega$. As $\{L_i\}_{i \in I_1} \subset \Omega$ и Ω is convex, that it has cone K_1, which is formed by the rays $\{L_i\}_{i\in I_1}$.

Let $I_2 = \{i \in I_1 : 0 < \beta_i < \infty\}$, $I_3 = \{i \in I : \beta_i = 0\}$, then $\forall x \in \Omega$ we have

$$x = \sum_{i\in I_1} \alpha_i e_i + \sum_{i \in I_2} \alpha_i e_i + \sum_{i\in I_3} \alpha_i e_i,$$

where $\alpha_i \ge 0 \quad \forall i \in I$.

We will show, that $\alpha_i = 0 \quad \forall i \in I_3$. If it is not, that $\exists i_0 \in I_3 : \alpha_{i_0} > 0$ and

$$\alpha_{i_0} e_{i_0} = x - (\sum_{i\in I_1} \alpha_i e_i + \sum_{i\in I_2} \alpha_i e_i + \sum_{i \in I_3 \setminus i_o} \alpha_i e_i) \subset x - K \subset \Omega - K \subset \overline{\Omega - K};$$

besides $\alpha_{i_0} e_{i_0} \in K$, i.e. $\alpha_{i_0} e_{i_0} \in \overline{\Omega - K} \cap K = \Omega$ (by normality Ω).

By definition $\beta_{i_0} = sup\{\beta \geqslant 0 : \beta e_{i_0} \in \Omega\} \ge \alpha_{i_0} > 0$, this contradicts the inclusion $i_0 \in I_3$.

We shown, that for any $x \in \Omega$ we have the equal $x = \sum\limits_{i\in I_1} \alpha_i e_i \in K_1$, i.e. $\Omega \subset K_1$. We have previously shown, that $K_1 \subset \Omega$, i.e. $\Omega = K_1$, but on the conditions of the proposition Ω can not be cone, thereby $I_2 \neq \varnothing$.

Now let K_2 is cone with forming $\{L_i\}_{i\in I_2}$ and $G = \Omega \cap K_2$.

The set G is finite, how the subset of finite set Ω.

Following from definition the execution for set G (0)-property with respect to subcone K_2. Check the normality (with respect to K_2) of the set G.

The inclusion $G \subset \overline{(G - K_2)}$ is obviously.

By definition $G \subset \Omega$, $K_2 \subset K$, so

$$\overline{G - K_2} \cap K_2 \subset \overline{\Omega - K_2} \cap K_2 \subset (\overline{\Omega - K} \cap K) \cap K_2 = \Omega \cap K_2 = G.$$

Of equality $G = \overline{G \cap K_2} \cap K_2$ and convexity G (as the intersection of convex sets Ω и K_2) we have normality G with respect to K_2.

Check the equality $\Omega = G + K_1$. As noted, $\forall x \in \Omega$ really the note

$$x = \sum_{i\in I_1} \alpha_i e_i + \sum_{i \in I_2} \alpha_i e_i = x_1 + x_2,$$

where $x_1 = \sum\limits_{i\in I_1} \alpha_i e_i$; $x_2 = \sum\limits_{i \in I_2} \alpha_i e_i$. Its true for $x_2 = x - x_1$ inclusion

$$x_2 \in \Omega - K_1 \subset (\Omega - K) \cap K_2 \subset (\Omega - K) \cap K \subset \overline{\Omega - K} \cap K = \Omega;$$

i.e. $x_2 \in \Omega \cap K_2 = G$, where from $x = x_1 + x_2 \in K_1 + G$, we proved the inclusion $\Omega \subset G + K_1$.

On the contrary, let it $x \in G + K_1$, i.e. $x = x_2 + x_1$, where $x_2 \in G$, $x_1 \in K$. If $x_1 = 0$, then $x = x_2 \in G = \Omega \cap K_2$, i.e. $x \in \Omega$.

Now let $x_1 \neq 0$ and $L = \{\alpha x_1, \alpha \geq 0\}$ is ray, which passes through the point x_1. As K_1 is cone, then $\lambda x_1 \in K_1 \quad \forall \lambda \geq 0$, $\lambda x_1 \in K \subset \Omega \quad \forall \lambda \geq 0$ and $x_2 \in G = \Omega \cap K_2 \subset \Omega$, by virtue of the convexity of the set Ω contains the points $\alpha x_2 + (1-\alpha)\lambda x_1$ $0 \leq \alpha < 1$ и $\lambda \geq 0$. We suppose that $\lambda = \frac{1}{1-\alpha}$ and will get, that $\alpha x_2 + x_1 \in \Omega$ for $0 \leq \alpha < 1$.

We consider the sequence

$$x^n = (1 - \frac{1}{n})x_2 + x_1, \quad x_n \in \Omega, \quad \text{т.к.} \quad 0 \leq 1 - \frac{1}{n} < 1 \quad \forall n,$$

then $\lim_n x^n = x_2 + x_1 \in \Omega$ by virtue of the convexity of the set Ω.

The proved inclusions $\Omega \subset G + K$ и $G + K \subset \Omega$ gave the equality $\Omega = G + K$.

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Көпбүйірлі конустың нормаль ішкі жиындарының құрылымы туралы

Мақалада нормаланған кеңістіктегі K көпбүйірлі конусының нормаль дөңес ішкі жиындарының құрылымы зерттелді. $\Omega \subset K$ ішкі жиынының нормальдығы (K конус мағынасында) $\overline{\Omega - K} \cap K = \Omega$ шартымен анықталды (жиын үстіндегі сызық топологиялық тұйықталу дегенді білдіреді). Конустың көпбүйірлілігі деп шеткі сәулелері болып табылатын сәулелер ақырлы санының канондық қабықшасын айтамыз. Нормаль жиындардың құрылымы геометриялық тұрғыдан зерттелді. Көпбүйірлі конустың кез келген Ω нормаль ішкі жиынын екі ішкі жиындардың қосындысына бөлуге болатындығы көрсетілді: оның бірі — шектелген нормаль жиын (кейбір K ішкі конус мағынасында), ал екіншісі — Ω жиынына тиісті K ішкі конусы (егер Ω шектелмеген болса, онда ол да шектелмеген).

Кілт сөздер: конус, нормаль, конустың қабығы, көпбүйірлі конус, сәулелер құрылымы, жабық жиынтығы, конус қалыпты ішкі жиыны.

Т.Х. Макажанова, А.С. Базылжанова, О.И. Ульбрихт

О строении нормальных подмножеств многогранного конуса

В статье исследовано строение нормальных выпуклых подмножеств многогранного конуса K в нормированном пространстве. Нормальность подмножества $\Omega \subset K$ (в смысле конуса K) определяется условием $\overline{\Omega - K} \cap K = \Omega$ (черта над множеством означает взятие топологического замыкания). Многогранность конуса означает, что он является конической оболочкой конечного числа лучей, являющихся крайними лучами. Исследовалось строение нормальных множеств с геометрической точки зрения. Показано, что всякое нормальное подмножество Ω многогранного конуса можно разбить на сумму двух подмножеств, одно из которых является ограниченным нормальным (в смысле некоторого подконуса в K) подмножеством, а второе — содержащимся во множестве Ω подконусом K (неограниченным, если Ω неограничено).

Ключевые слова: конус, нормаль, коническая оболочка, многогранный конус, образующие лучи, замкнутое множество, нормальное подмножество конуса.

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