UDC 53.25; 53.043

THE GRAVITATIONAL MODELS ON THE BASE OF TENSOR FIELD IN MINKOWSKI SPACE-TIME

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The class of classical gravitational tensor field in the Minkowski space-time models based on universal procedure of theories «free lagrangian" gravitational "dressing" has been developed. The procedure consists in operation of a "gravitational" operator on a "free lagrangian". The "gravitational" operator is specific for each model of the class. The common structure of those operators has been shown. The possible models are discussed. The construction of A. Einstein's theory of general relativity (GRT) is represented as an example of the used method. Mathematical realization of "co-variantization" principle for GRT has been given.

Keywords: gravitational tensor field, Minkowski space-time models, gravitational operator.

Gravitational "dressing"

Let's consider the gravitational field as a tensor field, described by the potential $\varphi_{\mu\nu}$ (a symmetrical tensor potential) in the Minkowski space-time, and bring in the simplest interaction, satisfying the principle of equivalency in the form of equality of the inertial and gravitational masses (Moshinsky's lagrangian [1])[2]:

$$^{(1)}L = ^{(0)}L - \varphi_{\mu\nu}\,^{(0)}T^{\mu\nu} \tag{1}$$

Here $^{(1)}L$ is a lagrangian, "dressed" by the gravitation in the first order on $\varphi_{\mu\nu}$, $^{(0)}L$ - *a free* lagrangian of theory ("substance" fields + non-self-acting gravitational field), $^{(0)}\mathrm{T}^{\mu\nu}$- is an energy impulse tensor of the "free" theory (IET-0). This is a traditional first step to construction of field formulation of GRT (Birkhoff [2], Thirring[3], Deser[4]).

Let us present $^{(0)}\mathrm{T}^{\mu\nu}$ as a variation derivative of the "free" functional operation according to Murkowski's metric ("metrical" TEI):

$$^{(0)}\mathrm{T}^{\mu\nu} = \frac{2}{\sqrt{-\eta}}\frac{\delta\left(\sqrt{-\eta}\;^{(0)}L\right)}{\delta\eta_{\mu\nu}} \tag{2}$$

Let us substitute (2) in (1):

$$^{(1)}L = {}^{(0)}L + \frac{2}{\sqrt{-\eta}}\varphi_{\mu\nu}\frac{\delta\left(\sqrt{-\eta}\,^{(0)}L\right)}{\delta\eta_{\mu\nu}}$$

Then multiply this equality by $\sqrt{-\eta}$ and move to a lagrangian's density:

$$N = \sqrt{-\eta}\;L$$

We obtain:

$$^{(1)}N = ^{(0)}N + 2\varphi_{\mu\nu}\frac{\delta\left(^{(0)}N\right)}{\delta\eta_{\mu\nu}} \tag{3}$$

Paid attention to the right part of equation (3), it is possible to suppose that there the expansion in series of some operator takes place (which converts the lagrangian density of “free” theory into the lagrangian’s density with interaction) into a series according to variation derivative:

$$2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}$$

Let us view the general expression for the lagrangian density, constructed by the given way

$$N = \left\{1 + 2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}} + +\kappa_2\left(2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}\right)^2 + \kappa_3\left(2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}\right)^3 + \ldots\right\}^{(0)}N \ , \quad (4)$$

where the coefficients $\kappa_i(i = 2,...,\infty)$ generally define an integro-differential “gravitational” operator:

$$N(n,\varphi) = \hat{F}\left(2\varphi\frac{\delta}{\delta\eta}0\right)^{(0)} N(\eta) \quad (5)$$

The most important property of this operator is its regularity (*a₁*,*a₂* are the constants)

$$\hat{F}(\varphi)(a_1N_1 + A_2N_2) = a_1\hat{F}(\varphi)N_1 + a_2\hat{F}(\varphi)N_2$$

Let us consider the lagrangian densities, which do not depend on the derivatives of metrical tensor on coordinates (or it is equivalent to on the Kristoffel’s connections). Let us present a “free” lagrangian density as a series of metrical tensor.

$$^{(0)}N = \sqrt{-\eta}\sum_n N_n\eta^n \ , \quad (6)$$

where the omitted space-time indexes and n mean a number of multipliers in summand. N_n is the expansion coefficients, which entirely defined by dynamic “substance” variables.

Substituting (6) into (5) and taking into consideration the operator $\hat{F}(\varphi)$ regularities we obtain:

$$N = \sum_n N_n\hat{F}(\varphi)\{\sqrt{-\eta}\eta^n\} \quad (7)$$

Thus, the only metrical coefficients of Lagrangian change to the lagrangian density of theory influenced by gravitational field while the “substance” coefficients do not change. Hence, the common form of laws of “substance” field dynamics under the action of gravity leaves unchangeable (gravitational «universality»). However, as we shall show that the gravitational «universality» does not result to the possibility of “geometrization” of any classes’ model.

Classes’ model “geometrization” possibility

Let us formulate the conditions of “geometrization” (transition to the pseudo-Rimanov time) of the lagrangian density (7). If it is possible in some case, the expression (7) presents the form of (6) with a metrical tensor:

$$N = \sqrt{-g}\sum_n N_n g^n \quad (8)$$

Having compared (7) and (8) it is clear that the following conditions will be correct in this case:

$$\sqrt{-g}g^{n} = \hat{F}(\varphi)\{\sqrt{-\eta}\eta^{n}\} \tag{9}$$

Therefore, the metrical coefficients for some n_1 and n_2 become the interconnected ones in a certain way. It is clear that the interconnection is impossible for arbitrary k_i in the operator $\hat{F}$.

Let us consider a lagrangian of massive scalar electrodynamics (in fact, the choice of a definite "substance" lagrangian is not important due to the gravitational "universality" shown above.)

$$L_M^{(0)} = \frac{1}{2}\eta^{\mu\nu}D_\mu\overline{\psi}D_\nu\psi - \frac{1}{2}m^2\overline{\psi}\psi + ie\eta^{\mu\nu}(\overline{\psi}D_\mu\psi - D_\mu\overline{\psi}\psi)A_\nu +$$
$$+ e^2\eta^{\mu\nu}\overline{\psi}\psi A_\mu A_\nu - \frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\alpha}F_{\mu\nu}F_{\alpha\beta}$$

It does not depend on Kristoffel's symbols and represents the expansion (6) in series.

The appropriate lagrangian density with gravitational interaction has the following form

$$N = -\frac{m^2}{2}\overline{\psi}\psi\hat{F}(\sqrt{-\eta}) + \{\frac{1}{2}\overline{D}_\mu\psi D_\nu\psi + ie(\overline{\psi}D_\mu\psi - D_\mu\overline{\psi}\psi)A_\nu +$$
$$+ e^2\overline{\psi}\psi A_\mu A_\nu\}\hat{F}(\sqrt{-\eta}\eta^{\mu\nu}) - -\frac{F_{\mu\nu}F_{\alpha\beta}}{4}\hat{F}(\sqrt{-\eta}\eta^{\mu\alpha}\eta^{\nu\beta}$$

Having compared this expression with the analogous "geometrizated" lagrangian density:

$$N = -\frac{m^2}{2}\overline{\psi}\psi(\sqrt{-g}) + \{\frac{1}{2}\overline{D}_\mu\psi D_\nu\psi + ie(\overline{\psi}D_\mu\psi - D_\mu\overline{\psi}\psi)A_\nu +$$
$$+ e^2\overline{\psi}\psi A_\mu A_\nu\}\sqrt{-g}g^{\mu\nu} - \frac{F_{\mu\nu}F_{\alpha\beta}}{4}\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}$$

we obtain:

$$\hat{F}(\varphi)\left\{\sqrt{-\eta}\right\} = \sqrt{-g}$$
$$\hat{F}(\varphi)\left\{\sqrt{-\eta}\eta^{\mu\nu}\right\} = \sqrt{-g}g^{\mu\nu} \tag{10}$$
$$\eta + 2\varphi$$

Let us introduce the denotations:

$${}^{0,0}Q = \hat{F}(\varphi)\left\{\sqrt{-\eta}\right\}$$
$${}^{1,0}Q^{\mu\nu} = \hat{F}(\varphi)\left\{\sqrt{-\eta}\eta^{\mu\nu}\right\} \tag{11}$$
$${}^{2,0}Q^{\mu\alpha\nu\beta} = \hat{F}(\varphi)\left\{\sqrt{-\eta}\eta^{\mu\alpha}\eta^{\nu\beta}\right\}$$

where the first preindex at metric coefficients Q means the number of multiplied metrical tensors, and the second one means the number of multiplied Kristoffel's symbols.

From (10) the "geometrization" conditions on the value Q follows:

$${}^{0,0}Q = (\det({}^{1,0}Q^{\mu\nu}))^{1/6}$$

$$ {}^{2,0}Q^{\mu\alpha\nu\beta} = \frac{{}^{1,0}Q^{\mu\alpha}.{}^{1,0}Q^{\nu\beta}}{{}^{0,0}Q} \tag{12} $$

The next quantity plays the role of a metrical tensor

$$ g^{\mu\nu} = \frac{{}^{1,0}Q^{\mu\nu}}{{}^{0,0}Q} \tag{13} $$

If the terms with n multipliers take place in lagrangian, at “geometrization “ after “dressing” by gravitational interaction they will be connected” by the following ratio:

$$ {}^{n,0}Q = \frac{\Pi_{i=1}^{n}\,{}^{1,0}Q_i}{{}^{0,0}Q^{n-1}} \tag{14} $$

where the space-time indexes and *n>1* are omitted.

For instance, let us assign the second expression of “geometrization” (12) conditions by means of the coefficients of “gravitational” operator.

$$ \left\{1+2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}}+k_2(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^2+k_3(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^3+\ldots\right\}\left\{\sqrt{-\eta}\eta^{\mu\alpha}\eta^{\nu\beta}\right\}= $$

$$ =\frac{\left\{1+2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}}+k_2(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^2+k_3(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^3+\ldots\right\}\sqrt{-\eta}\eta^{\mu\alpha}}{\left\{1+2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}}+k_2(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^2+k_3(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^3+\ldots\right\}\left\{\sqrt{-\eta}\right\}}* $$

$$ *\left\{1+2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}}+k_2(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^2+k_3(2\varphi_{\sigma\lambda}\frac{\delta}{\delta\eta_{\sigma\lambda}})^3+\ldots\right\}\left\{\sqrt{-\eta}\eta^{\nu\beta}\right\} $$

Comparing the coefficients at the same terms in the left and right parts, we obtain:

$$ k_2=\frac{1}{2}=\frac{1}{2!} \quad k_3=\frac{1}{6}=\frac{1}{3!}. \tag{15} $$

κ_n are the exponent expansion coefficients in Taylor’s series .At any other coefficients in a “gravitational” operator, the models do not allow “geometrization”.

Models’ metric coefficients interrelation

It is important fact that different metrical coefficients in the lagrangians, which are similar to the lagrangian of massive scalar electrodynamics are interrelated owing to the following relations from differential geometry (the used techniques of differentiation on metric tensor is given in Supplement A):

$$ \frac{\partial(\sqrt{-\eta})}{\partial\eta_{\mu\nu}} = \frac{\sqrt{\eta}}{2}\eta^{\mu\nu} \tag{16} $$

$$ \frac{\partial(\sqrt{\eta}\eta^{\alpha\beta})}{\partial\eta_{\mu\nu}} + \frac{\partial(\sqrt{-\eta}\eta^{\alpha\nu})}{\partial\eta_{\mu\beta}} = \sqrt{\eta}\eta^{\mu\alpha}\eta^{\nu\beta} \tag{17} $$

$$\frac{\partial(\sqrt{\eta}\eta^{\alpha\beta})}{\partial\eta_{\mu\nu}}-2\frac{\partial(\sqrt{-\eta}\eta^{\sigma\nu}\eta^{\beta\lambda})}{\partial\eta_{\mu\alpha}}-2\frac{\partial(\sqrt{-\eta}\eta^{\sigma\nu}\eta^{\alpha\lambda})}{\partial\eta_{\mu\beta}}-\frac{\partial(\sqrt{-\eta}\eta^{\beta\nu}\eta^{\alpha\lambda})}{\partial\eta_{\mu\sigma}}-$$
$$-2\frac{\partial(\sqrt{-\eta}\eta^{\nu\lambda}\eta^{\alpha\beta})}{\partial\eta_{\mu\sigma}}-2\frac{\partial(\sqrt{-\eta}\eta^{\beta\nu}\eta^{\alpha\sigma})}{\partial\eta_{\mu\lambda}}-2\frac{\partial(\sqrt{-\eta}\eta^{\sigma\nu}\eta^{\alpha\beta})}{\partial\eta_{\mu\lambda}}+2\frac{\partial(\sqrt{-\eta}\eta^{\mu\beta}\eta^{\beta\lambda})}{\partial\eta_{\alpha\nu}}+$$
$$+2\frac{\partial(\sqrt{-\eta}\eta^{\alpha\lambda}\eta^{\beta\sigma})}{\partial\eta_{\mu\sigma}}-2\frac{\partial(\sqrt{-\eta}\eta^{\alpha\beta}\eta^{\mu\lambda})}{\partial\eta_{\nu\sigma}}+2\frac{\partial(\sqrt{-\eta}\eta^{\mu\nu}\eta^{\sigma\lambda})}{\partial\eta_{\alpha\beta}}+\frac{\partial(\sqrt{-\eta}\eta^{\mu\sigma}\eta^{\nu\lambda})}{\partial\eta_{\alpha\beta}}+$$
$$+3\frac{\partial(\sqrt{-\eta}\eta^{\mu\beta}\eta^{\alpha\nu})}{\partial\eta_{\sigma\lambda}}=\frac{9}{2}\sqrt{\eta}\eta^{\mu\nu}\eta^{\sigma\lambda} \quad \text{etc.} \qquad (18)$$

Let us denote the metrical coefficients in the lagrangian as following:

$$^{0,0}q=\sqrt{-\eta}$$
$$^{1,0}q^{\mu\nu}=\sqrt{-\eta}\eta^{\mu\nu}$$
$$^{2,0}q^{\mu\nu\beta}=\sqrt{-\eta}\eta^{\mu\alpha}\eta^{\nu\beta}$$
$$^{3,0}q^{\mu\nu\alpha\beta\varsigma\alpha}=\sqrt{-\eta}\eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\sigma\alpha}$$

Then, from (16),(17),(18) it follows:

$$^{1,0}q^{\mu\nu}=2\frac{\partial(^{0,0}q)}{\partial\eta_{\mu\nu}} \qquad (19)$$

$$^{2,0}q^{\mu\iota\alpha\beta}=-\frac{\partial(^{1,0}q^{\alpha\beta})}{\partial\eta_{\mu\nu}}-\frac{\partial(^{1,0}q^{\alpha\nu})}{\partial\eta_{\mu\beta}} \qquad (20)$$

$$^{3,0}q^{\mu\nu\alpha\beta\sigma\lambda}=\frac{2}{9}\left\{\frac{\partial(^{2,0}q^{\alpha\sigma\beta\lambda})}{\partial\eta_{\mu\nu}}-2\frac{\partial(^{2,0}q^{\sigma\beta\nu\lambda})}{\partial\eta_{\mu\alpha}}-2\frac{\partial(^{2,0}q^{\sigma\alpha\nu\lambda})}{\partial\eta_{\mu\beta}}-\frac{\partial(^{2,0}q^{\alpha\beta\nu\lambda})}{\partial\eta_{\mu\sigma}}\right\} -$$
$$\frac{2}{9}\left\{2\frac{\partial(^{2,0}q^{\alpha\nu\beta\lambda})}{\partial\eta_{\mu\sigma}}-2\frac{\partial(^{2,0}q^{\alpha\beta\sigma\nu})}{\partial\eta_{\mu\lambda}}-2\frac{\partial(^{2,0}q^{\alpha\sigma\beta\nu})}{\partial\eta_{\mu\lambda}}+2\frac{\partial(^{2,0}q^{\mu\beta\sigma\lambda})}{\partial\eta_{\lambda\nu}}+\frac{\partial(^{2,0}q^{\alpha\beta\sigma\lambda})}{\partial\eta_{\mu\sigma}}\right\}+ \qquad (21)$$
$$\frac{2}{9}\left\{-2\frac{\partial(^{2,0}q^{\mu\alpha\beta\lambda})}{\partial\eta_{\nu\sigma}}+2\frac{\partial(^{2,0}q^{\mu\sigma\nu\lambda})}{\partial\eta_{\alpha\beta}}+\frac{\partial(^{2,0}q^{\mu\nu\sigma\lambda})}{\partial\eta_{\alpha\beta}}+3\frac{\partial(^{2,0}q^{\mu\alpha\nu\beta})}{\partial\eta_{\sigma\lambda}}\right\} \quad \text{and} \quad \text{etc.}$$

Thus, all "free" metrical coefficients in the considered lagrangian are finally expressed in term of the "zero" coefficient $^{0,0}q$. It is worth to emphasize, that this interrelation of "free" metrical coefficients does not relate to gravitation at all.

As an assumption it is possible to regard the idea of considering such lagrangians as a result of some operator's effect in the form of derivatives expansion on metrical tensor with coefficients as functions of "substance" fields on prelagrangian $^{0,0}q$.

Let us consider the "dressed" (by gravitation) metrical coefficients:

$$^{0,0}Q=\hat{F}(\varphi)(^{0,0}q)$$
$$^{1,0}Q^{\mu\nu}=\hat{F}(\varphi)(^{1,0}q^{\mu\nu})$$
$$^{2,0}Q^{\mu\nu\alpha\beta}=\hat{F}(\varphi)(^{2,0}q^{\mu\nu\alpha\beta})$$

From (19),(20),(21) we obtain:

$$^{1,0}Q^{\mu\nu} = \hat{F}(\varphi)(2\frac{\partial(^{0,0}q)}{\partial\eta_{\mu\nu}})$$

$$^{2,0}Q^{\mu\nu\alpha\beta} = -\hat{F}(\varphi)(\frac{\partial(^{1,0}q^{\alpha\beta})}{\partial\eta_{\mu\nu}}) - \hat{F}(\varphi)(\frac{\partial(^{1,0}q^{\alpha\nu})}{\partial\eta_{\mu\beta}}) \quad \text{and etc.}$$

Here, the question arises: how are the “dressed” metrical coefficients interrelated? Is it, perhaps, in the same way as “free” ones? Therefore, we should discuss the questions with commutation of “gravitational” operator and derivative on metrical tensor.

Taking into consideration that the operator $\hat{F}(\varphi)$ is one of the functions of variational derivative it becomes clear that everything comes to the research of commutation of variational derivative and the derivative on metric tensor. However, these operators commutate and therefore the commutation follows:

$$\hat{F}(\varphi)\frac{\partial}{\partial\eta_{\mu\nu}} = \frac{\partial}{\partial\eta_{\mu\nu}}\hat{F}(\varphi)$$

Thus, the relations (19), (20), and etc. are still true for the ”dressed” (by gravitation) metrical coefficients and the construction of model of lagrangian leads to the research of the effect of “gravitational” operator on “zero” metrical coefficient.

The construction of Einstein general relativity theory (GRT)

We have already studied the case when one of the classes’ models allows the “geometrization”. Using the coefficients from (15) for “gravitational” operator, we obtain:

$$N = \left\{1 + 2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}} + \frac{1}{2!}\left(2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}\right)^2 + \frac{1}{3!}\left(2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}\right)^3 + ...\right\} {}^{(0)}N \quad , \tag{22}$$

where the sum of terms of series in the square brackets will give obviously the operator exponent:

$$N = \exp\left\{2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}\right\} {}^{(0)}N \tag{23}$$

The lagrangian density ${}^{(0)}N$ is a function of metrical tensor $\eta_{\mu\nu}$ and its first derivatives on coordinates (Kristoffel’s symbols). The exponent (obtained above) makes a functional metric shift by double gravitational tensor:

$$N(\eta\varphi) = {}^{(0)}\mathrm{N}(\eta + 2\varphi) \tag{24}$$

Thus, we have the theory, which is analogous to “free” one, but in space-time with the metric $\eta + 2\varphi$, that is in some pseudo-Rimanov space-time. The received model is equivalent to the Einstein formulation of GRT field ([2-4]).

Let us show the derivation of the Einstein gravitational field equations in the given model. As it is known (Landau-Lifshitz [5]), the given equations follow from Gilbert lagrangian (scalar curvature of space-time). Let the “free” (that is without gravitational interrelation) lagrangian density of all fields except gravitational one be equal to ${}^{(0)}N_{mat}(\eta)$. We add the scalar of the Minkowski space-time curvature to this density:

$$ {}^{(0)}N(\eta) = {}^{(0)}N_{mat}(\eta) + \sqrt{-\eta}R(\eta) \quad (25) $$

Now, let us apply the operational exponent on this density: (23)

$$ N(\eta,\varphi) = \exp\left\{2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}\right\}{}^{(0)}N(\eta) = \exp\left\{2\varphi_{\mu\nu}\frac{\delta}{\delta\eta_{\mu\nu}}\right\}({}^{(0)}N_{mat}(\eta) + \sqrt{-\eta}R(\eta)) $$

The exponent effect leads to "co-variantization" of free density that is to transition to space-time with the metric $g_{\mu\nu} = \eta_{\mu\nu} + 2\varphi_{\mu\nu}$

$$ N(g) = N_{mat}(g) + \sqrt{-g}R(g) \quad (26) $$

Thus, we have the lagrangian density (GRT), where the Einstein equations follow.

Conclusion

The procedure of gravitational "dressing" of a free lagrangian, expressed by the formulas (4) and (5) allows creating subsequent models of gravitational field as a tensor field in the Minkowski space-time. Thus, all created models satisfy to the most part of the classical experimental tests (GRT) as they coincide with it in the first order in the series of the "gravitational" operator. However, the most known test refers to exception. It is a shift of Mercury perihelion, which requires higher corrections. Thus, our main task is to search the leading principle, which let us to form theory. Further, we shall try to discuss this principle

References:

1. Moshinky M., «On the interactions of Birkoff's gravitational field with the electromagnetic and pair fields», 1950, Phys. Rev., **Vol. 80**, p. 514.
2. Birkhoff G.D. «Flat space-time and gravitation», 1944, Proc. Nat. Acad. Sci., **Vol.30**, p. 324.
3. Thirring W., "An alternative approach to the theory of gravitation", 1961, Phys. Rev., **Vol. 123**, p. 384.
4. Deser S., «Self-interaction and gauge invariance», 1970, Gen. Rel. Grav., **Vol. 1**, p. 9.
5. Landau L.D., Lifshitz E.M. «The classical Theory of Fields» **Vol.2**, 1988 «Science», p. 345.

[1] The "free" lagrangian means a lagrangian without interaction of gravitational field with fields of "substance" and itself .
[2] Here, the system of units has been chosen, where the speed of light c and the Plank constant are equal to 1.

Supplement A

In the article, the following rule of differentiation on metric tensor component is accepted:

$$ \frac{\partial\eta_{\alpha\beta}}{\partial\eta_{\mu\nu}} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right) \quad (A.1) $$

Using (A.1) together with the basic property of the metrical tensor $\eta_{\mu\alpha}\eta^{\alpha\nu} = \delta^{\nu}_{\mu}$ it is easy to obtain the formula for the derivatives from contra-variant metrical tensor components

$$ \frac{\partial\eta^{\alpha\beta}}{\partial\eta_{\mu\nu}} = -\frac{1}{2}\left(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\beta}\eta^{\beta\mu}\right) \quad (A.2) $$

In addition, the great role in the article belongs to the formula

$$ \frac{\partial\left(\sqrt{-\eta}\right)}{\partial\eta_{\mu\nu}} = \frac{\sqrt{-\eta}}{2}\eta^{\mu\nu} \quad (A.3) $$