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S.E. Aitzhanov[1,2,*], Zh. Tileuberdi[1], G. Sanat[1]

[1] *Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[2] *International Information Technology University, Almaty, Kazakhstan*
(E-mail: Aitzhanov.Serik81@gmail.com, zhainagan.jaina@gmail.com, gibadat_97-10@mail.ru)

Solvability of an initial-boundary value problem for a nonlinear pseudoparabolic equation with degeneration

This article is devoted to the solvability of degenerate nonlinear equations of pseudoparabolic type. Such problems appear naturally in physical and biological models. The article aims to study the solvability in the classes of regular solutions of (all derivatives generalized in the sense of S.L. Sobolev included in the equation) initial-boundary value problems for differential equations. For the problems under consideration, We have found conditions on parameters ensuring the existence of solutions and we have proved existence and uniqueness theorems. The main method for proving the solvability of boundary value problems is the regularization method.

Keywords: pseudo parabolic equations, degenerate equations, boundary value problems, nonlinear equations, solvability, uniqueness.

Introduction

In the modern theory of partial differential equations, an important place is occupied by the study of degenerate hyperbolic and elliptic equations, as well as equations of mixed type. The increased interest in this class of equations is explained both by the great theoretical significance of the obtained results and by their numerous applications in gas dynamics, hydrodynamics, in the theory of infinitesimal bending of the surface, in the momentless theory of shells, in various branches of mechanics of continuous media, acoustics, and in the theory of electron scattering and many other areas of expertise. Degenerate equations are a good model for physical and biological processes. Such equations have become an actual formulation and solution of various boundary value problems. Consequently, degenerate equations are currently the subject of fundamental research by many mathematicians.

Boundary value problems for pseudo parabolic equations were investigated in the works of D. Colton [1], A.M. Nakhushev [2], A.I. Kozhanov [3], M.S. Salakhitdinov [4], T.D. Dzhuraev [5], and others.

One of the important sections of the theory of partial differential equations is the formulation and study of well-posed boundary value problems for degenerate parabolic equations of the second, third, and higher orders.

Boundary value problems for second-order degenerate parabolic equations are considered in the works of M. Gevrey [6], O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Uralczeva [7], C.D. Pagani, G. Talenti [8], Yu.P. Gorkov [9].

In this article, we consider boundary value problems for differential equations of the following type:

$$\varphi(t)u_t - \nu\Delta u - \chi\Delta u_t + |u|^{p-2}u + c(x,t)u = f(x,t) \;\; (x \in \Omega \subset R^n,\, n \geq 3,\; 0 < t < T) \qquad (1)$$

where $\nu = const > 0$ and $f(x,t)$ is the external force. In these equations, the function $\varphi(t)$ and $\chi(t)$ can arbitrarily change sign on the segment $[0,T]$, and it can vanish on subsets of the segment $[0,T]$ of positive measure.

In article [3] I.A. Kozhanov and E.E. Maczievskaya in a cylindrical domain $Q = \Omega \times (0,T)$ $(0 < T < +\infty$, $\Omega \subset R^n$ – bounded area with smooth border $\Gamma)$ considered the solvability of a boundary value problem for differential equations:

$$\varphi(t)\, u_t + \psi(t)\, \Delta u + c(x,t)\, u = f(x,t)\, (x \in \Omega \subset R^n, 0 < t < T)\,. \qquad (2)$$

*Corresponding author.
E-mail: Aitzhanov.Serik81@gmail.com

In articles [10], [11] for equation (2), a statement of the first boundary value problem is proposed, and the existence of its generalized solutions is proved.

G. Fichera [10] considered an equation of the following type:

$$Lu = a^{ij}u_{x_i x_j} + b^i u_{x_i} + cu = f(x,t)$$

and for the first boundary value problem the existence of its generalized solutions is proved.

O.A. Olejinik and E.V. Radkevich [11] proved the existence of generalized solutions to the first boundary value problem for the following type of equation:

$$Lu = a^{kj}u_{x_k x_j} + b^k u_{x_k} + c(x)u = f(x).$$

I.A. Kozhanov [12] proved the uniqueness of solutions to the first boundary value problem for the following type of equation:

$$u_{tt} + \alpha(t)\frac{\partial}{\partial t}(\Delta u) + Bu = f(x,t).$$

The purpose of this work is to study the solvability of the first boundary value problem for doubly degenerate differential equations (1) in classes of regular solutions — solutions that have all derivatives generalized in the sense of Sobolev entering the equation.

In [13], A. Benaissa and Ch. Aichi considered a one-dimensional degenerate wave equation with a boundary control condition of fractional derivative type

$$u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 \text{ in } (0,1)\times(0,\infty), \quad (3)$$

where the coefficient a is a positive function on $[0,1]$ but vanishes at zero. The degeneracy of (3) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\le 1}\frac{x|a'(x)|}{a(x)}.$$

The researchers pointed out that the problem is not uniformly stable by a spectrum method and they studied the polynomial stability using the semigroup theory of linear operators.

In [14], the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x(x,t))_x + \alpha u_t(x,t) + \beta y(x,t) = 0, \ 0<x<1, \ t>0, \\ (a(x)u_x)(0) = k_1 u_t(0,t), \ t>0, \\ (a(x)u_x)(1) = -k_2 u_t(1,t), \ t>0, \end{cases}$$

where

$$\begin{cases} \alpha \ge 0, \ \beta > 0, \ k_1, \ k_2 \ge 0, \ k_1 + k_2 \ne 0, \\ a \in W^{1,\infty}(0,1), \ a(x) \ge a_0, \ \forall x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

In [15], the authors considered a biharmonic regularization to the following nonlinear degenerate elliptic equation:

$$\begin{aligned} Qu = \textstyle\sum_{i=1}^d \left[\partial_{x_i}\left(\sum_{j=1}^d a_{ij}(x,Du)\partial_{x_j}u + b_i(x,u,Du)\right) + c_i(x,u,Du)\partial_{x_i}u\right] + d(x)u = \\ = f + \textstyle\sum_{i=1}^d \partial_{x_i}g^i, \ x \in \Omega \subset R^d, \ Du = \nabla u = (\partial_{x_1},...,\partial_{x_d}), \end{aligned}$$

where the coefficients will be specified later. By degenerate ellipticity, we imply that the coefficients a_{ij}, $i,j = 1,...,d$, satisfy degenerate ellipticity conditions

$$0 \le \lambda(x,p)|\xi|^2 \le a_{ij}(x,p)\xi_i\xi_j, \ x \in \Omega, \ p \in R^d,$$

for all $\xi = (\xi_1,...,\xi_d) \in R^d\backslash 0$. Under appropriate assumption on the coeffcients, we prove that a sequence of biharmonic regularization to a nonlinear degenerate elliptic equation with possibly rough coeffcients preserves certain regularity as the approximation parameter tends to zero. In order to obtain the result, they introduced a generalization of the Chebyshev inequality. They also presented numerical example.

In [16], the author considered degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities:

$$\begin{cases} \partial_t b^j(u) - \nabla \cdot (a(x) \nabla u_t)^j - \nabla \cdot d^j(t, x, u, \nabla u) + M^j(u) = f^j(u), \\ u^j = 0 \text{ on } (0,T) \times \partial\Omega, \\ b^j(u(0,x)) = b^j(u_0(x)) \text{ in } \Omega, \end{cases}$$

where the memory operator M is defined by

$$\left\langle M^j(t)(u), v^j \right\rangle = \int_\Omega \int_0^t K^j(t,s)\, g^j(s, x, \nabla u(s,x))\, ds \nabla v^j(t,x)\, dx$$

for all functions $u,\ v \in L^p\left(0, T; H_0^{1,p}(\Omega)^l\right)$, for almost all $t \in (0,T)$.

The existence of solutions of degenerate quasilinear pseudoparabolic equations, where the term $\partial_t u$ is replaced by $\partial_t b(u)$, with memory terms and quasilinear variational inequalities is shown. The existence of solutions of equations is proved under the assumption that the nonlinear function b is monotone and a gradient of a convex, continuously differentiable function. The uniqueness is proved for Lipschitz continuous elliptic parts. The existence of solutions of quasilinear variational inequalities is proved under stronger assumptions, namely, the nonlinear function defining the elliptic part is assumed to be a gradient and the function b to be Lipschitz continuous.

Statement of the problem

Let $\Omega \subset R^n,\ n \geq 3$ is a bounded domain with the smooth border $\partial\Omega$, Q_T is a cylinder $\Omega \times (0,T)$ of finite height T, $S = \partial\Omega \otimes (0,T)$ is a side boundary. Further, let $\nu > 0$, χ, p are constants, $\varphi(t), c(x,t)$ and $f(x,t)$ be the given functions defined at $t \in [0,T],\ x \in \bar{\Omega}$, L is a differential operator whose action on a given $w(x,t)$ is determined by the equality

$$Lw = \varphi(t) w_t - \nu \Delta w - \chi \Delta w_t + |w|^{p-2} w + c(x,t) w$$

where $2 < p < 4$, Δ is the Laplace operator in the variables $x_1, x_2, ..., x_n$.

Boundary problem I. Find a function $u(x,t)$ that is a solution of the equation:

$$Lu = f(x,t) \tag{4}$$

in the $Q_T = \Omega \times (0,T)$ and such that condition:

$$u|_S = 0, \tag{5}$$

$$u(x,0) = 0,\ x \in \Omega.$$

Boundary problem II. Find a function $u(x,t)$ that is a solution of equation (4) in the $Q_T = \Omega \times (0,T)$ and such that conditions (5) and

$$u(x,0) = u(x,T) = 0,\ x \in \Omega.$$

Solvability of boundary value problems I-II

Theorem 1. Let the conditions

$$\varphi(t) \in C^1[0,T], c(x,t) \in C^2(\bar{Q}_T); \tag{6}$$

$$2c(x,t) - \varphi'(t) \geq c_1 > 0,\ \ 2c(x,t) + \varphi'(t) \geq c_2 > 0 \text{ at } (x,t) \in \bar{Q}_T; \tag{7}$$

$$\varphi(0) \leq 0,\ \ \varphi(T) > 0; \tag{8}$$

$$f(x,t) \in W_2^{1,1}(Q_T), \quad f(x,t) = 0 \text{ at } (x,t) \in S, \quad f(x,0) = 0. \tag{9}$$

Then there is a unique solution $u \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem I.

Proof. For the proof, we use the regularization method. Let ε be a positive number. Let L_ε denote the differential operator whose action on a given function $w\,(x,t)$ is determined by the equality

$$L_\varepsilon w = \varepsilon \Delta w_{tt} + Lw.$$

Consider a boundary value problem: Find a function $w\,(x,t)$ that is a solution of the equation

$$L_\varepsilon w = f\,(x,t) \tag{10}$$

in the $Q_T = \Omega \times (0,T)$ and with conditions (5) and

$$u(x,0) = u_t(x,T) = 0,\; x \in \Omega. \tag{11}$$

Note that the first priori estimate is valid

$$\varepsilon \left\|\nabla u_t\right\|^2_{2,Q_T} + \left\|u\right\|^2_{2,Q_T} + \left\|\nabla u\right\|^2_{2,Q_T} + \left\|u\right\|^p_{p,Q_T} \le c_3. \tag{12}$$

To prove this estimate, it suffices to analyze the equality

$$\int_{Q_T} L_\varepsilon u u dxdt = \int_{Q_T} f u dxdt$$

using the conditions of the theorem (6), (7), (8), (9), and Young's inequality.

Consider the following equation:

$$-\int_Q L_\varepsilon u \Delta u dxdt = -\int_Q f \Delta u dxdt.$$

Let us write this equation by integrating by parts:

$$\begin{array}{l}\varepsilon \left\|\Delta u_t\right\|^2_{2,Q_T} + \frac{1}{2}\int_\Omega \varphi(T)\left|\nabla u(x,T)\right|^2 dx - \\ -\frac{1}{2}\int_{Q_T} \varphi'(t)\left|\nabla u\right|^2 dxdt + \frac{\chi}{2}\int_\Omega \left|\Delta u(x,T)\right|^2 dx + \\ +\nu \int_{Q_T} \left|\Delta u\right|^2 dxdt + (p-1)\int_{Q_T} \left|u\right|^{p-2} |\nabla u|^2 dxdt + \int_{Q_T} c(x,t)|\nabla u|^2 dxdt = \\ -\int_{Q_T} \nabla c(x,t) u \nabla u dxdt - \int_{Q_T} f \Delta u dxdt.\end{array} \tag{13}$$

Let us estimate right-hand side of the equation (13):

$$\left|-\int_{Q_T} \nabla c(x,t) u \nabla u dxdt\right| \le c_4 \left\|\nabla u\right\|_{2,Q_T} \left\|u\right\|_{2,Q_T} \le \left\|\nabla u\right\|^2_{2,Q_T} + \frac{c_4}{4}\left\|u\right\|^2_{2,Q_T}.$$

$$\left|\int_Q f \Delta u dxdt\right| \le \frac{\nu}{2}\left\|\Delta u\right\|^2_{2,Q_T} + \frac{1}{2\nu}\left\|f\right\|^2_{2,Q_T}.$$

Substituting the obtained inequalities into equation (13) and taking into account the conditions of the theorem (6)–(9), we obtain the second priori estimate

$$\varepsilon \left\|\Delta u_t\right\|^2_{2,Q_T} + \int_{Q_T} \left|\Delta u\right|^2 dxdt + \int_{Q_T} \left|u\right|^{p-2} |\nabla u|^2 dxdt \le C_5. \tag{14}$$

Now, let us show that at conditions (9), the solutions $u \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\, W_2^2(\Omega)) \cap$ $\cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem (10), (5), (11) will satisfy the estimates uniform by ε.

In the next step, consider equality

$$\int_Q L_\varepsilon u \Delta u_{tt} dxdt = -\int_Q f \Delta u_{tt} dxdt.$$

This equality is easily transformed to form

$$
\begin{array}{l}
\varepsilon\left\|\Delta u_{tt}\right\|_{L^2(Q_T)}^2+\frac{1}{2}\int_{\Omega}\varphi(0)\left|\nabla u_t(x,0)\right|^2dx+\\
+\frac{\chi}{2}\int_{\Omega}\left|\Delta u_t(x,0)\right|^2dxdt+\nu\int_{Q_T}\left|\Delta u_t\right|^2dxdt+\frac{1}{2}\int_{Q_T}(2c(x,t)+\varphi'(t))\left|\nabla u_t\right|^2dxdt=\\
=-\int_{Q_T}(c_t(x,t)\nabla u\nabla u_t+\nabla c(x,t)u_t\nabla u_t+\nabla c_t(x,t)u\nabla u_t)dxdt-\\
-(p-1)\int_{Q_T}|u|^{p-2}u_t\Delta u_tdxdt+\int_{Q_T}\nabla f_t\nabla u_tdxdt.
\end{array}.
$$

Let us estimate to $\int_{Q_T}|u|^{p-2}u_t\Delta u_tdxdt$:

$$
\begin{array}{l}
\left|\int_{Q_T}|u|^{p-2}u_t\Delta u_tdxdt\right|\le\left\|\Delta u_t\right\|_{2,Q_T}\left\|u_t\right\|_{\frac{2n}{n-2},Q_T}\left\|u\right\|_{(p-2)n,Q_T}^{p-2}\le\\
\le\frac{\nu}{2}\left\|\Delta u_t\right\|_{2,Q_T}^2+\frac{c_1}{8}\left\|\nabla u_t\right\|_{2,Q_T}^2+C_6.
\end{array}
$$

Using the conditions of the theorem (6)–(9), we obtain from this that solution $u\left(x,t\right)$ of the boundary value problem (10), (5), (11) satisfies the estimate

$$
\varepsilon\left\|\Delta u_{tt}\right\|_{L^2(Q_T)}^2+\int_{Q_T}\left|\Delta u_t\right|^2dxdt+\int_{Q_T}\left|\nabla u_t\right|^2dxdt\le C_7. \tag{15}
$$

Estimates (12), (14) and (15) are already enough for choosing a sequence converging to the solution of boundary value problem I.

Let $\{\varepsilon_l\}_{l=1}^{\infty}$ be a sequence of positive numbers converging to 0. We denote by $u_l\left(x,t\right)$ the solution to boundary value problem (10), (5), (11) for $\varepsilon=\varepsilon_l$. For the sequence $\{u_l\left(x,t\right)\}_{l=1}^{\infty}$ for $\varepsilon=\varepsilon_l$, the priori estimates (12), (14) and (15) hold. It follows from these estimates and the reflexive property of the Hilbert space that there exists a subsequence $\{u_{l_k}\left(x,t\right)\}_{k=1}^{\infty}$ and a function $u\left(x,t\right)$ such that

$$\varepsilon_{l_k}\to 0,$$

$$u_{l_k}\left(x,t\right)\to u\left(x,t\right)\ \text{ in } L_2\left(Q_T\right)\ \text{ weakly},$$

$$\nabla u_{l_k t}\left(x,t\right)\to\nabla u_t\left(x,t\right)\ \text{ in } L_2\left(Q_T\right)\ \text{ weakly},$$

$$\Delta u_{l_k}\left(x,t\right)\to\Delta u\left(x,t\right)\ \text{ in } L_2\left(Q_T\right)\ \text{ weakly},$$

$$\Delta u_{l_k t}\left(x,t\right)\to\Delta u_t\left(x,t\right)\ \text{ in } L_2\left(Q_T\right)\ \text{ weakly},$$

$$\varepsilon_{l_k}\Delta u_{l_k tt}\left(x,t\right)\to 0\ \text{ in } L_2\left(Q_T\right)\ \text{ weakly},$$

converges for $k\to\infty$. Obviously, the limit function $u\left(x,t\right)$ will belong to the space $u\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(Q_t)$, and that it is a solution to the problem I. □

The study of the solvability of boundary value problem II in classes $u\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(Q_t)$ is carried out in the whole similarly to the study of the solvability of boundary value problem I. The regularization method is used again, the operator L_ε is again used as a regularizing operator. The difference is that in the regularizing problem at $t=0$ and $t=T$ there are conditions

$$u(x,0)=u(x,T)=0,x\in\Omega.$$

Theorem 2. Let conditions (6), (7) and (9), and conditions

$$\varphi(0)>0,\,\varphi(T)<0$$

also hold. Then there is a unique solution $u\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(0,T;\,W_2^2(\Omega))\cap\overset{0}{W}{}_2^1(\Omega)$, $u_t\in L_2(Q_t)$ of the boundary value problem II.

The uniqueness of a solution

Theorem 3. Let conditions (6)–(9) be satisfied. Then $u(x,t)$ the solution of the boundary value problem I is a unique.

Proof. To prove the uniqueness of the equation suppose that problem has two solutions: $u_1(x,t)$ and $u_2(x,t)$. Then their difference $\vartheta(x,t) = u_1(x,t) - u_2(x,t)$ satisfies condition

$$\vartheta(x,0) = \vartheta_t(x,T) = 0,\ x \in \Omega.$$

Then, (10) is written in a form

$$\varepsilon\Delta\vartheta_{tt} + \varphi(t)\vartheta_t + |u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 - \nu\Delta\vartheta - \chi\Delta\vartheta_t + c(x,t)\vartheta = 0.$$

Consider the following equation:

$$\int_{Q_T} \Big(\varepsilon\Delta\vartheta_{tt} + \varphi(t)\vartheta_t + |u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 - \nu\Delta\vartheta - \\ -\chi\Delta\vartheta_t + c(x,t)\vartheta)\,\vartheta dxdt = 0.$$

For any $p > 0$, the inequality holds

$$|(|u_1|^p u_1 - |u_2|^p u_2)(u_1 - u_2)| \geq c\,|u_1 - u_2|^{p+2}.$$

Let us write this equation by integrating by parts

$$\varepsilon\,\|\nabla\vartheta_t\|^2_{2,Q_T} + \frac{c_1}{2}\int_{Q_T}\vartheta^2 dxdt + \\ +\frac{\nu}{2}\int_{Q_T}|\nabla\vartheta|^2 dxdt + c\int_{Q_T}|\vartheta|^p \leq 0$$

In this case, we come to an equality

$$\vartheta = 0 \Rightarrow u_1 - u_2 = 0 \Rightarrow u_1 = u_2.$$

So, we proved the uniqueness of a solution.

Statement of the second problem

Let $\Omega \subset R^n$, $n \geq 3$ is a bounded domain with the smooth border $\partial\Omega$, Q_T is a cylinder $\Omega \times (0,T)$ with a finite height T, $S = \partial\Omega \otimes (0,T)$ is a side boundary. Further, let $\nu > 0$, χ, q be constants, $\varphi(t)$, $c(x,t)$ and $f(x,t)$ be the given functions defined at $t \in [0,T]$, $x \in \bar{\Omega}$, L is a differential operator whose action on a given $w(x,t)$ is determined by the equality

$$Lw = \varphi(t)w_t - \nu\Delta w - \chi\Delta w_t - |\nabla w|^q + c(x,t)w$$

where $0 < q < 1$, Δ is the Laplace operator in the variables $x_1, x_2, ..., x_n$.

Boundary problem III. Find a function $u(x,t)$ that is a solution of the equation

$$Lu = f(x,t) \tag{16}$$

in the $Q_T = \Omega \times (0,T)$ and with the conditions

$$u|_S = 0, \tag{17}$$

$$u(x,0) = 0,\ x \in \Omega.$$

Boundary problem IV. Find a function $u(x,t)$ that is a solution of the equation (16) in the $Q_T = \Omega \times (0,T)$ and with condition (17), and condition

$$u(x,0) = u(x,T) = 0,\ x \in \Omega.$$

Solvability of boundary value problems I-II

Theorem 4. Let conditions (6), (7) and (9), and conditions

$$\varphi(0) \leq 0,\ \varphi(T) > 0$$

also hold. Then there is a unique solution $u \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem III.

Theorem 5. Let conditions (6), (7) and (9), and conditions

$$\varphi(0) > 0,\ \varphi(T) < 0$$

also hold. Then there is a unique solution $u \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(0,T;\, W_2^2(\Omega)) \cap \overset{0}{W}{}_2^1(\Omega)$, $u_t \in L_2(Q_t)$ of the boundary value problem III.

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С.Е. Айтжанов[1,2], Ж. Тілеуберді[1], Ғ. Санат[1]

[1] *Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан;*
[2] *Халықаралық ақпараттық технологиялар университеті, Алматы, Қазақстан*

Азғындалған сызықты емес псевдопараболалық теңдеу үшін бастапқы-шеттік есептің шешілімділігі

Мақала псевдопараболалық типтегі азғындалған сызықты емес теңдеулердің шешілімділігіне арналған. Мұндай проблемалар физиканың және биологияның әр түрлі модельдерінде туындайды. Мақаланың мақсаты — дифференциалдық теңдеулер үшін шекті есептердің (С.Л.Соболев мағынасында жалпыланған барлық туындыларды қоса алғанда) регулярлы шешімдер класындағы шешімділікті зерттеу. Қарастырылып отырған есептің шешілуіне кепілдік беретін, параметрлерге шарттар табылған және қарастырылған есептер үшін шешімнің бар және жалғыздық теоремалары дәлелденген. Шектік есептердің шешімділігін дәлелдеудің негізгі әдісі регуляризация әдісі болады.

Кілт сөздер: псевдопараболық теңдеулер, азғындалған теңдеулер, шеттік есептер, сызықты емес теңдеулер, шешімділік, жалғыздық.

С.Е. Айтжанов[1,2], Ж. Тилеуберди[1], Г. Санат[1]

[1] *Казахский национальный университет имени аль-Фараби, Алматы, Казахстан;*
[2] *Международный университет информационных технологий, Алматы, Казахстан*

Разрешимость начально-краевой задачи для нелинейного псевдопараболического уравнения с вырождением

Статья посвящена разрешимости вырожденных нелинейных уравнений псевдопараболического типа. Такие задачи естественно возникают в физических и биологических моделях. Целью статьи является исследование разрешимости в классах регулярных решений (включающих в уравнение все производные, обобщенные по С.Л.Соболеву) краевых задач для дифференциальных уравнений. Для рассматриваемых задач авторами найдены условия на параметры, гарантирующие, что задача имеет решение. Кроме того, доказаны теоремы существования и единственности. В качестве основного метода доказательства разрешимости краевых задач выбран метод регуляризации.

Ключевые слова: псевдопараболические уравнения, вырожденные уравнения, краевые задачи, нелинейные уравнения, разрешимость, единственность.

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