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Differential inequality and non-oscillation of fourth order differential equation

The oscillatory theory of fourth order differential equations has not yet been developed well enough. The results are known only for the case when the coefficients of differential equations are power functions. This fact can be explained by the absence of simple effective methods for studying such higher order equations. In this paper, the authors investigate the oscillatory properties of a class of fourth order differential equations by the variational method. The presented variational method allows to consider any arbitrary functions as coefficients, and our main results depend on their boundary behavior in neighborhoods of zero and infinity. Moreover, this variational method is based on the validity of a certain weighted differential inequality of Hardy type, which is of independent interest. The authors of the article also find two-sided estimates of the least constant for this inequality, which are especially important for their applications to the main results on the oscillatory properties of these differential equations.

Keywords: fourth order differential equation, oscillation, non-oscillation, variational principle, weighted inequality, space.

1 Introduction

Let $I = (0, \infty)$. Let v be a positive function twice differentiable on I and u be a non-negative function continuous on I.

We consider the following fourth order differential equation

$$(v(t)y''(t))'' - \lambda u(t)y(t) = 0,\ t \in I, \tag{1}$$

where $\lambda > 0$ is a real number.

The oscillatory properties of equation (1) have not yet been studied sufficiently. The obtained results are mainly the case where v or u are power functions. There are also results where equation (1) has been studied by its reduction to a Hamiltonian system and application of the Riccati technique using unknown fundamental solutions of the system. The development of the oscillation theory of equation (1) is given in the works [1–3], and references therein. For more details, we also refer to the monograph [4].

One more method to investigate the oscillatory properties of (1) is the variational method. This method is based on the fact that non-oscillation of equation (1) is equivalent to the validity of a certain second order differential inequality, which allows to obtain non-oscillation conditions in terms of the functions v and u. However, the known results on this differential inequality are not suitable for using them by this method. In this paper, under some assumptions on the function v in neighborhoods of zero and infinity, we find suitable characterizations for the validity of this second order differential inequality, and then apply them to obtain non-oscillation conditions of equation (1).

Let us note that the study of differential equations of fourth and higher orders by the variational method began in the works [5] and [6] under assumptions on the function v different from those presented here. More precisely, characterizations of the corresponding inequality depend on the number of zero boundary conditions at each endpoint of the interval, where this inequality is considered. This number, in turn, depends on assumptions on v. In the work [6], the corresponding second order inequality is studied on the interval $I_T = (T, \infty)$, $T > 0$,

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under assumptions on v providing the existence of exactly two boundary conditions both at T. In this paper, we consider the interval I and assumptions on v are such that they also provide the existence of exactly two boundary conditions but one condition is at zero while the second is at infinity. In addition, the equation considered in [6] contains the differential operation $D_r^2 y(t) = \frac{d}{dt} r(t) \frac{dy(t)}{dt}$, where r is a positive function sufficiently time continuously differentiable on I. This operation becomes $y''(t)$ for $r = 1$.

This paper is organized as follows. Section 2 contains all the auxiliary statements necessary to prove the main results. Section 3 establishes the validity of the required second order differential inequality. In Section 4, on the basis of this inequality we find non-oscillation conditions of equation (1). Section 5 contains an example.

2 Auxiliary statements

Let $0 \le a < b \le \infty$. From the work [7], we have the following lemma.

Lemma A. (i) The inequality

$$\int_a^b u(x) \left(\int_a^x f(t)\, dt \right)^2 dx \le C \int_a^b v(t) f^2(t) dt \tag{2}$$

holds if and only if

$$A^+ = \sup_{a<z<b} \int_z^b u(x) dx \int_a^z v^{-1}(t)\, dt < \infty,$$

in addition, $A^+ \le C \le 4A^+$, where C is the best constant in (2).

(ii) The inequality

$$\int_a^b u(x) \left(\int_x^b f(t)\, dt \right)^2 dx \le C \int_a^b v(t) f^2(t) dt \tag{3}$$

holds if and only if

$$A^- = \sup_{a<z<b} \int_a^z u(x) dx \int_z^b v^{-1}(t)\, dt < \infty,$$

in addition, $A^- \le C \le 4A^-$, where C is the best constant in (3).

Denote by $W_{2,v}^2 \equiv W_{2,v}^2(I)$ the space of functions $f : I \to \mathbb{R}$ twice differentiable on the interval I, for which the norm

$$\|f\|_{W_{2,v}^2} = \|f''\|_{2,v} + |f'(1)| + |f(1)| \tag{4}$$

is finite, where $\|g\|_{2,v} = \left(\int_0^\infty v(t) g^2(t) dt \right)^{\frac{1}{2}}$.

Let $C_0^\infty(I)$ be the set of finitely supported functions infinitely differentiable on the interval I. By the conditions on the function v we have that $C_0^\infty(I) \subset W_{2,v}^2(I)$. Denote by $\mathring{W}_{2,v}^2 \equiv \mathring{W}_{2,v}^2(I)$ the closure of the set $C_0^\infty(I)$ with respect to norm (4).

For $f \in W_{2,v}^2$ we assume that $\lim\limits_{t\to 0^+} f(t) = f(0)$ and $\lim\limits_{t\to\infty} f'(t) = f'(\infty)$.

Denote by $W_{2,v}^2(0,1)$ and $W_{p,v}^2(1,\infty)$ the contraction sets of functions from $W_{2,v}^2$ on the intervals $(0,1]$ and $[1,\infty)$, respectively.

Assume that

$$P_0(0,1) = \{C\chi_{(0,1)}(t) :\ C \in \mathbb{R}\},$$

$$P_1(1,\infty) = \{C\chi_{(1,\infty)}(t) t :\ C \in \mathbb{R}\},$$

$$L_0 W = \{f \in W_{p,v}^2 :\ f(0) = 0\},$$

$$R_1 W = \{f \in W_{p,v}^2 :\ f'(\infty) = 0\}.$$

From the work [8], we have one more statement.

Lemma B. (*i*) If $v^{-1} \notin L_1(0,1)$ and $t^2 v^{-1}(t) \in L_1(0,1)$, then $\mathring{W}^2_{2,v}(0,1) = L_0 W$; in addition, $W^2_{2,v}(0,1) = \mathring{W}^2_{2,v}(0,1) \dot{+} P_0(0,1)$.
(*ii*) If $v^{-1} \in L_1(1,\infty)$ and $t^2 v^{-1}(t) \notin L_1(1,\infty)$, then $\mathring{W}^2_{2,v}(1,\infty) = R_1 W$; in addition, $W^2_{2,v}(1,\infty) = \mathring{W}^2_{2,v}(1,\infty) \dot{+}$ $\dot{+} P_1(1,\infty)$.
Here the sign $\dot{+}$ means the direct sum of subspaces.

3 Differential inequality

Function $y : I \to \mathbb{R}$ is called a solution of equation (1) if it is four times continuously differentiable on the interval and satisfies equation (1) for all $t > 0$.

Equation (1) is called oscillatory at infinity (at zero) if for any $T > 0$ there exists a solution of this equation having more than one double zero to the right (to the left) of T. Otherwise, equation (1) is called non-oscillatory.

Let us consider the following second order differential inequality

$$\lambda \int_T^\infty u(t)|f(t)|^2 dt \le C_T \int_T^\infty v(t)|f''(t)|^2 dt, \quad f \in \mathring{W}^2_{2,v}(T,\infty). \tag{5}$$

In the work [9], on the basis of the variational principle [10] there was established the following lemma.

Lemma C. Let C_T be the least constant in (5). The equation (1) is non-oscillatory at infinity if and only if for some $T > 0$ we have that $0 < C_T \le 1$.

If we consider the inequality

$$\lambda \int_0^T u(t)|f(t)|^2 dt \le C_T \int_0^T v(t)|f''(t)|^2 dt, \quad f \in \mathring{W}^2_{2,v}(0,T), \tag{6}$$

we can write the statement similar to Lemma C for non-oscillation of equation (1) at zero.

Lemma C yields that non-oscillation of equation (1) depends on the constant C_T in (5) and (6). Therefore, we need to find the value of C_T or at least estimate it from above and below.

We investigate equation (1) under the following conditions at zero and infinity:

$$\int_0^1 v^{-1}(t)dt = \infty, \quad \int_0^1 t^2 v^{-1}(t)dt < \infty, \quad \int_1^\infty v^{-1}(t)dt < \infty, \quad \int_1^\infty t^2 v^{-1}(t)dt = \infty. \tag{7}$$

Under these assumptions we consider the inequality

$$\lambda \int_0^\infty u(t)|f(t)|^2 dt \le C_0 \int_0^\infty v(t)|f''(t)|^2 dt, \quad f \in \mathring{W}^2_{2,v}(I). \tag{8}$$

Let

$$E_1 = \sup_{z>0} \int_z^\infty u(t)dt \int_0^z s^2 v^{-1}(s)\, ds,$$

$$E_2 = \sup_{z>0} \int_0^z t^2 u(t)dt \int_z^\infty v^{-1}(s)\, ds,$$

$$E = \lambda \max\{E_1, E_2\}.$$

Theorem 1. Let (7) hold. Then inequality (8) holds if and only if $E < \infty$; in addition, $E \le C_0 \le 8E$, where C_0 is the best constant in (8).

Proof. From condition (7) and Lemma B, it follows that

$$\mathring{W}^2_{2,v}(I) = \{f \in W^2_{2,v} : f(0) = 0;\ f'(\infty) = 0\} \equiv LRW.$$

Hence, for $f \in \mathring{W}^2_{2,v}(I)$ we have $f(t) = \int\limits_0^t f'(s)ds$ and $f'(s) = -\int\limits_s^\infty f''(x)dx$. Then $f(t) = -\int\limits_0^t \int\limits_s^\infty f''(x)dxds = -\int\limits_0^t \int\limits_s^t f''(x)dxds - t\int\limits_t^\infty f''(x)dx = -\int\limits_0^t xf''(x)dx - t\int\limits_t^\infty f''(x)dx$. Using this relation, we get

$$\lambda \int\limits_0^\infty u(t)|f(t)|^2 dt = \int\limits_0^\infty u(t) \left| \int\limits_0^t xf''(x)dx + t\int\limits_t^\infty f''(x)dx \right|^2 dt$$

$$\leq 2\lambda \int\limits_0^\infty u(t) \left| \int\limits_0^t xf''(x)dx \right|^2 dt + 2\lambda \int\limits_0^\infty t^2 u(t) \left| \int\limits_t^\infty f''(x)dx \right|^2 dt. \quad (9)$$

The latter gives that if

$$\int\limits_0^\infty u(t) \left| \int\limits_0^t xf''(x)dx \right|^2 dt \leq C_1 \int\limits_0^\infty v(t)|f''(t)|^2 dt \quad (10)$$

and

$$\int\limits_0^\infty t^2 u(t) \left| \int\limits_t^\infty f''(x)dx \right|^2 dt \leq C_2 \int\limits_0^\infty v(t)|f''(t)|^2 dt \quad (11)$$

with the least constants C_1 and C_2, respectively, then $C_0 \leq 2\lambda \max\{C_1, C_2\}$, where C_0 is the least constant in (8). From Lemma A we have that $C_1 \leq 4E_1$ and $C_2 \leq 4E_2$. Therefore,

$$C_0 \leq 8E. \quad (12)$$

Now, assuming $f'' \geq 0$ in (9), we get that if (8) holds, then (10) and (11) also hold and $C_0 \geq \lambda \max\{C_1, C_2\}$. From Lemma A, it follows that $E_1 \leq C_1$ and $E_2 \leq C_2$, which together with (12) yields that $E \leq C_0 \leq 8E$. The proof of Theorem 1 is complete.

4 Non-oscillation of equation (1)

Theorem 2. Let (7) hold. Then equation (1) is non-oscillatory at infinity and zero if

$$\sup_{z>0} \int\limits_z^\infty u(t)dt \int\limits_0^z s^2 v^{-1}(s)\, ds \leq \frac{1}{8\lambda}, \quad (13)$$

$$\sup_{z>0} \int\limits_0^z t^2 u(t)dt \int\limits_z^\infty v^{-1}(s)\, ds \leq \frac{1}{8\lambda}. \quad (14)$$

Proof. From assumption (7) it follows that the function v^{-1} is non-singular at the point $T > 0$, i.e., for any finite $N > T$ we have $\int\limits_T^N v^{-1}(t)\, dt < \infty$. Therefore, for any $f \in \mathring{W}^2_{2,v}(T, \infty)$ we get $f(T) = f'(T) = 0$ and

$$\mathring{W}^2_{2,v}(T, \infty) = \{f \in W^2_{2,v}(T, \infty) : f(T) = f'(T) = f'(\infty) = 0\} \equiv L^2RW.$$

We expand the function $f \in L^2RW$ by zero on the interval $(0, T)$, i.e., we assume that $f(t) = 0$ for $0 < t < T$. This gives that $f \in \mathring{W}^2_{2,v}(I)$. Therefore, $LRW \supset L^2RW$. Then

$$C_0 = \sup_{f \in LRW} \frac{\int\limits_0^\infty u(t)|f(t)|^2 dt}{\int\limits_0^\infty v(x)|f''(x)|^2 dx} \geq \sup_{f \in L^2RW} \frac{\int\limits_0^\infty u(t)|f(t)|^2 dt}{\int\limits_0^\infty v(x)|f''(x)|^2 dx}$$

$$= \sup_{f \in L^2RW} \frac{\int\limits_T^\infty u(t)|f(t)|^2 dt}{\int\limits_T^\infty v(x)|f''(x)|^2 dx} = C_T. \qquad (15)$$

From (13) and (14), it follows that $E \leq \frac{1}{8\lambda}$. Hence, by Theorem 1 we have that $0 < C_0 \leq 1$. Therefore, due to (15), we get that $0 < C_T \leq 1$. Thus, by Lemma C, it follows that equation (1) is non-oscillatory at infinity.

Now, we turn to non-oscillation of equation (1) at zero. In this case, at the point $T > 0$ we have that $f(T) = f'(T) = 0$ for $f \in \mathring{W}^2_{2,v}(0,T)$ and

$$\mathring{W}^2_{2,v}(0,T) = \{f \in W^2_{2,v}(0,T) : f(0) = f(T) = f'(T) = 0\} \equiv LR^2W.$$

We expand the function $f \in \mathring{W}^2_{2,v}(0,T)$ by zero on the interval (T,∞) and get $LRW \supset LR^2W$. Arguing as above in (15), we establish that from $0 < C_0 \leq 1$ it follows $0 < C_T \leq 1$. By Theorem 1 from (13) and (14) we have $0 < C_0 \leq 1$. Thus, Lemma C written for inequality (6) yields that equation (1) is non-oscillatory at zero. The proof of Theorem 2 is complete.

5 Example

As an example, let us consider the following differential equation

$$(t^\alpha y''(t))'' - \lambda t^{-\beta} y(t) = 0, \ t \in I, \qquad (16)$$

where $\lambda > 0$ is a real number. Assume that $1 < \alpha, \beta < 3$. It is easy to see that the function $v^{-1}(t) = t^{-\alpha}$ satisfies condition (7). Then

$$E_1(z) = \int\limits_z^\infty t^{-\beta} dt \int\limits_0^z s^{2-\alpha}\, ds = \frac{z^{1-\beta}}{\beta - 1} \cdot \frac{z^{3-\alpha}}{3-\alpha} = \frac{z^{4-\alpha-\beta}}{(\beta-1)(3-\alpha)}.$$

$E_1 = \sup\limits_{z>0} E_1(z) < \infty$ if and only if $4 - \alpha - \beta = 0$ that is $\beta = 4 - \alpha$. Then $E_1 = \frac{1}{(\beta-1)(3-\alpha)} = \frac{1}{(3-\alpha)^2}$. Similarly, we can find that $E_2 = \frac{1}{(\alpha-1)^2}$. By Theorem 2 equation (16) is non-oscillatory at infinity and zero if $E_1 = \frac{1}{(3-\alpha)^2} \leq \frac{1}{8\lambda}$ and $E_2 = \frac{1}{(\alpha-1)^2} \leq \frac{1}{8\lambda}$. Therefore, equation (16) is non-oscillatory at infinity and zero if $\lambda \leq \frac{1}{8} \min\{(\alpha-1)^2, (3-\alpha)^2\}$. Thus, we can write the following proposition.

Proposition. Let $1 < \alpha < 3$ and $\beta = 4 - \alpha$. Then equation (16) is non-oscillatory at infinity and zero if $\lambda \leq \frac{1}{8} \min\{(\alpha-1)^2, (3-\alpha)^2\}$.

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Дифференциалдық теңсіздік және төртінші ретті дифференциалдық теңдеудің тербелімсіздігі

Төртінші ретті дифференциалдық теңдеулердің тербелімділік теориясы жеткілікті түрде дамымаған. Нәтижелер дифференциалдық теңдеулер коэффициенттері дәрежелік функциялары болған жағдайда ғана белгілі болады. Бұл фактіні жоғары дәрежелі теңдеулерді зерттеудің қарапайым тиімді әдістерінің болмауымен түсіндіруге болады. мақалада төртінші ретті дифференциалдық теңдеулер класының тербелмелі қасиеттері вариациялық әдіспен зерттелген. Ұсынылған вариациялық әдіс теңдеулер коэффициенттері кез келген функция болуы ретінде қарастыруға мүмкіндік береді және негізгі нәтижелер олардың нөлге және шексіздікке жақын шекаралық әрекеттеріне байланысты. Сонымен қатар, бұл вариациялық әдіс тәуелсіз қызығушылық тудыратын Харди типті салмақты дифференциалдық теңсіздігінің негізділігіне талқыланған. Осы теңсіздік үшін ең кіші константаның екі жақты бағалауы табылған, бұл олардың осы дифференциалдық теңдеулердің тербелмелік қасиеттерінің негізгі нәтижелеріне қолданылуы үшін ерекше маңызды.

Кілт сөздер: төртінші ретті дифференциалдық теңдеу, тербелімділік, тербелімсіздік, вариациялық принцип, салмақты теңсіздік, кеңістік.

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Дифференциальное неравенство и неосцилляторность дифференциального уравнения четвертого порядка

Теория осцилляций дифференциальных уравнений четвертого порядка недостаточно хорошо изучена. Известны результаты только для случая, когда коэффициенты дифференциальных уравнений являются степенными функциями. Этот факт можно объяснить отсутствием простых эффективных методов для изучения уравнений высокого порядка. В статье исследованы осцилляционные свойства одного класса дифференциальных уравнений четвертого порядка вариационным методом. Представленный вариационный метод позволяет рассматривать любые произвольные функции в качестве коэффициентов, а основные результаты зависят от их граничного поведения в окрестностях нуля и бесконечности. Более того, этот вариационный метод основан на выполнении некоторого весового дифференциального неравенства типа Харди, представляющего самостоятельный интерес. Авторами найдены двусторонние оценки наименьшей константы для этого неравенства, которые особенно важны для их приложений к основным результатам по осцилляторности рассматриваемых дифференциальных уравнений.

Ключевые слова: дифференциальное уравнение четвертого порядка, осцилляторность, неосцилляторность, вариационный принцип, весовое неравенство, пространство.

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