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ON A PARTICULAR SECOND KIND VOLTERRA INTEGRAL EQUATION WITH A SPECTRAL PARAMETER

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Abstract: We study the solvability questions for a particular second kind Volterra integral equation with a spectral parameter λ arising in the theory of boundary value problems for the spectrally loaded parabolic equations in unbounded domains when the order of a derivative in a loaded summand agrees with that in the differential part of the equation.

Keywords: Volterra integral equation, particular kernel, Riemann problem, index

Introduction. Despite the fact that the pair of conjugate integral equations studied in this article consists of second kind Volterra equations with a spectral parameter λ, the method of successive approximations [1] for their solving is not applicable, since the kernels of the integral operators have singularities. We demonstrate that the index of the integral operator in question is nonpositive and establish its dependence on the spectral parameter. Similar integral equations (with some other kernels) examined earlier in [2, 3] arise in the theory of inverse problems and the theory of loaded differential equations [4–6].

1. Statements of the problems. We examine the solvability questions for the following pair of conjugate integral equations: ($\mathbb{R}_+ = (0,\infty)$):

$$k_\lambda\varphi \equiv (I-\lambda k)\varphi \equiv \varphi(t) - \lambda\int\limits_0^t k(t-\tau)\varphi(\tau)\,d\tau = f(t), \quad 0<\tau<t<\infty, \tag{1}$$

$$k_\lambda^*\psi \equiv (I-\bar\lambda k^*)\psi \equiv \psi(t) - \bar\lambda\int\limits_t^\infty k(\tau-t)\psi(\tau)\,d\tau = g(t), \quad 0<t<\tau<\infty, \tag{1*}$$

where

$$k(t) = \frac{1}{2\sqrt{\pi}t^{3/2}}\exp\left(-\frac{1}{4t}\right), \quad t>0.$$

Note that the peculiarity of (1) and (1*) is the fact that k and k^* possess the property

$$\lim_{t\to+0}\int\limits_0^t k(t-\tau)\,d\tau = 0; \qquad \int\limits_t^\infty k(\tau-t)\,d\tau = 1 \quad \forall t>0.$$

Thereby the norm of k^* is equal to 1 and the method of successive approximations is not applicable to (1*).

The data and solutions to (1) and (1*) meet the conditions

$$\lambda\in\mathbb{C}; \quad f(t),\varphi(t)\in L_1(\mathbb{R}_+); \quad g(t),\psi(t)\in L_\infty(\mathbb{R}_+), \tag{2}$$

where f and g are given functions.

Transform (1) and (1*) using the corresponding one-sided functions defined for φ, f, ψ, and g by the expressions

$$l_+(\theta) = \begin{cases} l(\theta), & \text{if } \theta>0,\\ 0, & \text{if } \theta\le 0,\end{cases} \qquad l_-(\theta) = \begin{cases} 0, & \text{if } \theta\ge 0,\\ -l(\theta), & \text{if } \theta<0,\end{cases}$$
$$l(\theta) = l_+(\theta) - l_-(\theta), \quad \theta\in\mathbb{R};$$

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and for k, by the expression

$$k_+(\theta) = \begin{cases} k(\theta), & \text{if} \quad \theta > 0, \\ 0, & \text{if} \quad \theta \le 0, \end{cases} \quad k_-(\theta) = \begin{cases} 0, & \text{if} \quad \theta \ge 0, \\ k(-\theta), & \text{if} \quad \theta < 0. \end{cases} \tag{3}$$

In this case (1) and (1*) imply that

$$(I - \lambda \mathbf{k}_+)\varphi_+ \equiv \varphi_+(t) - \lambda \int_{-\infty}^{+\infty} k_+(t-\tau)\varphi_+(\tau)\, d\tau = f_{2+}(t) + \varphi_-(t), \quad t \in \mathbb{R}, \tag{4}$$

$$(I - \lambda \mathbf{k}_-)\psi_+ \equiv \psi_+(t) - \bar{\lambda} \int_{-\infty}^{+\infty} k_-(t-\tau)\psi_+(\tau)\, d\tau = g_{2+}(t) + \psi_-(t), \quad t \in \mathbb{R}. \tag{4*}$$

If $t > 0$ then (4) and (4*) coincide with (1) and (1*), respectively, and below we show that solutions to (1) and (1*) do not depend on their extensions to the negative semiaxis, i.e. on $\varphi_-(t)$ and $\psi_-(t)$.

2. Solving the integral equation (4*). Applying the Fourier transform to (4*), we infer

$$\mathbf{\Psi}^+(s) - \bar{\lambda}\mathbf{K}^-(s)\mathbf{\Psi}^+(s) = \mathbf{G}_2^+(s) + \mathbf{\Psi}^-(s), \quad s \in \mathbb{R}, \tag{5}$$

where the capitals denote the corresponding Fourier images; moreover,

$$\mathbf{K}^-(s) = \exp[-(1 + i\,\mathrm{sign}(s))\sqrt{|s|/2}].$$

This function admits the analytic continuation $\mathbf{K}^-(z) = \exp\{-\sqrt{iz}\}$ to the whole complex plane $z = s + i\sigma$ with the cut along the positive imaginary axis. Moreover, there exist an analytic function $\mathbf{\Psi}^+(z)$ defined in the upper half-plane $z = s + i\sigma$, $\sigma \ge 0$, and an analytic function $\mathbf{\Psi}^-(z)$ defined in the lower half-plane $z = s + i\sigma$, $\sigma \le 0$, whose traces on the real axis $\sigma = 0$ are equal to $\mathbf{\Psi}^+(s)$ and $\mathbf{\Psi}^-(s)$, respectively.

If

$$\mathbf{A}_\lambda^*(s) \equiv 1 - \bar{\lambda}\mathbf{K}^-(s) \ne 0 \quad \forall s \in \mathbb{R} \tag{6}$$

then (5) gives rise the Riemann problem (see [7–9])

$$\mathbf{\Psi}^+(s) = [\mathbf{A}_\lambda^*(s)]^{-1}\mathbf{\Psi}^-(s) + \bar{\lambda}\mathbf{R}_\lambda^-(s)\mathbf{G}_2^+(s) + \mathbf{G}_2^+(s), \quad s \in \mathbb{R}, \tag{7}$$

where $\mathbf{R}_\lambda^-(s) = \mathbf{K}^-(s)/\mathbf{A}_\lambda^*(s)$. The coefficient $\mathbf{R}_\lambda^-(s)$ in (7) has the analytic continuation $\mathbf{R}_\lambda^-(z)$ to the plane of the variable $z = s + i\sigma$ with the cut along the positive imaginary axis $\mathbf{R}_\lambda^-(z)$ with simple poles at

$$z_k = s_k + i\sigma_k = -2(\arg\lambda + 2k\pi)\log|\lambda| - i[\log^2|\lambda| - (\arg\lambda + 2k\pi)^2], \quad k \in \mathbb{Z}, \tag{8}$$

which are the zeros of the function $\mathbf{A}_\lambda^*(z)$ and lie on the parabola

$$z = s + i\left(\frac{s^2}{4\log^2|\lambda|} - \log^2|\lambda|\right), \quad s \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}. \tag{9}$$

Obviously, the vertex of (9) lies on the imaginary axis of the plane z and moves up and down in dependence on $|\lambda|$; the branches of this parabola are directed upwards.

Clarify some properties of $\mathbf{A}_\lambda^*(z)$. The function $\exp\{-\sqrt{iz}\}$ is multivalued. Choose the branch of this function such that $\mathrm{Re}\sqrt{iz_k} > 0$. To this aim, in the plane z we make the cut along the positive imaginary axis. We obtain

$$\lambda\exp(-\sqrt{iz}) = |\lambda|\exp[i\arg\lambda - \sqrt{|z|\exp i(\pi/2 + \arg z + 2n\pi)}]$$
$$= |\lambda|\exp[i\arg\lambda - \sqrt{|z|}\exp i(\pi/4) + n\pi + (\arg z)/2]$$
$$= |\lambda|\exp[-\sqrt{|z|}\cos(\pi/4 + n\pi + (\arg z)/2) + i\arg\lambda - \sqrt{|z|}\sin(\pi/4 + n\pi + (\arg z)/2)],$$

where $n = 0, 1$. In order the inequality $\mathrm{Re}\{iz_k\} > 0$ to be valid, it is necessary to assume that $n = 1$, since in this case $\cos[(5\pi)/4 + (\arg z)/2] > 0$ for $\pi/2 < \arg z < (5\pi)/2$. Hence, the function $\mathbf{A}^*_\lambda(z)$ is uniquely determined on the complex plane with the cut along the positive imaginary axis and has zeros only for $|\lambda| > 1$. Thereby $\mathbf{A}^*_\lambda(z)$ does not vanish for $|\lambda| < 1$.

We rewrite (6) in terms of an appropriate complex parameter λ. The formula (8) for the roots z_k implies that (6) is fulfilled if and only if the imaginary parts of these roots do not vanish, i.e., if $\log^2|\lambda| - (\arg\lambda + 2k\pi)^2 \neq 0$ for all $k \in \mathbb{Z}$. The last condition together with the inequality $|\lambda| > 1$ is equivalent to the requirement

$$|\lambda| \neq \exp(|\arg\lambda + 2k\pi|) \quad \forall k \in \mathbb{Z}. \tag{10}$$

If $|\lambda| < 1$ then, obviously, $\mathbf{A}^*_\lambda(z)$ does not vanish on the complex plane $z = s + i\sigma$ with the cut along the positive imaginary semiaxis since $|\exp(\sqrt{iz})| > 1$. Indeed, for the function $\mathbf{A}^*_\lambda(z)$ to have zeros it is necessary that $|\lambda| = |\exp(\sqrt{iz})|$. The last equality is impossible under our assumptions.

But if $|\lambda| = 1$ then the equation $|\lambda| = |\exp(\sqrt{iz})|$ with the complex variable λ has the only solution $\lambda = 1$ which corresponds to the value $z = 0$.

The lines described by the equations $|\lambda| = \exp(|\arg\lambda + 2k\pi|)$ divide the complex plane of the parameter λ into pairwise disjoint domains D_m, $m = 0, 1, 2, \ldots$, as follows:

$$\begin{gathered} D_{2n} = \{D_n^{(1)} \cap D_n^{(2)}\} \setminus \bigcup_{k=-1}^{2n-1} D_k, \quad D_{-1} = \phi, \\ D_{2n+1} = \{D_n^{(1)} \cup D_n^{(2)}\} \setminus \bigcup_{k=0}^{2n} D_k, \end{gathered} \tag{11}$$

where $D_n^{(1)} = \{\lambda : |\lambda| < \exp[(2n+1)\pi - \arg\lambda]\}$ and $D_n^{(2)} = \{\lambda : |\lambda| < \exp[2n\pi + \arg\lambda]\}$, $n = 0, 1, 2, \ldots$. Denote the exterior parts of the boundaries ∂D_m of D_m, $m = 0, 1, 2, \ldots$, by Γ_m, $m = 0, 1, 2, \ldots$. Observe that the boundary of D_0 has only the exterior part $\Gamma_0 = \partial D_0$ while the boundaries ∂D_m of the remaining domains D_m consist of the exterior part Γ_m and the interior part Γ_{m-1}, i.e., $\partial D_m = \Gamma_{m-1} \cup \Gamma_m$. Furthermore, $\Gamma_{m-1} \cap \Gamma_m = (-1)^m \exp\{m\pi\}$, i.e., the exterior part Γ_m and the interior part Γ_{m-1} of the boundary ∂D_m of D_m have one common point lying on the real axis of the complex plane λ.

Moreover, we can find that $\lambda \in \Gamma_m$, $m = 0, 1, 2, \ldots$, if and only if there exists at least one point $\tilde{s}$ such that $\mathbf{A}^*_\lambda(\tilde{s}) = 0$.

Let $|\lambda| > 1$. In accord with (9), $\mathbf{A}^*_\lambda(z)$ can have finitely many zeros of the form (8) in the lower half-plane, where

$$-N_1 \leq k \leq N_2, \quad N_1 = [(\log|\lambda| + \arg\lambda)/(2\pi)], \quad N_2 = [(\log|\lambda| - \arg\lambda)/(2\pi)]; \tag{12}$$

here $[a]$ stands for the entire part of a; moreover, we assume that the entire part of a negative number is equal to zero. Indeed, (12) follows from the condition $\mathrm{Re}\{-iz_k\} \leq 0$ for the roots (8) (in accord with (2) for a solution $\psi(t)$ from the class of essentially bounded functions). Hence, the inequality $(2\pi k + \arg\lambda)^2 < \log^2|\lambda|$ yields (12).

The Riemann problem (7) has positive index $\varkappa^*(\lambda)$ which is equal to the number of zeros (with multiplicity counted) of the function $\mathbf{A}^*_\lambda(z)$ in the lower half-plane, i.e.,

$$\varkappa^*(\lambda) = \mathrm{Ind}\{[\mathbf{A}^*_\lambda(z)]^{-1}\} = -\mathrm{Ind}\{\mathbf{A}^*_\lambda(z)\} = N_1 + N_2 + 1 > 0. \tag{13}$$

Note that in view of (8) and (9) the index of the Riemann problem is equal to $\varkappa^*(\lambda) = 0$ for $|\lambda| < 1$. Let $\sum_{k=-N_1}^{N_2} \frac{c_k}{z-z_k}$ be the principal part of the Laurent expansion of the function $[\mathbf{A}^*_\lambda(z)]^{-1}\mathbf{\Psi}^-(z)$ in the powers of $z - z_k$, $k = -N_1, \ldots, 0, \ldots, N_2$. In this case

$$\chi(z) \equiv \frac{\mathbf{\Psi}^-(z)}{\mathbf{A}^*_\lambda(z)} - \sum_{k=-N_1}^{N_2} \frac{c_k}{z - z_k}$$

is a function whose Fourier preimage vanishes for $t \in \mathbb{R}_+$. Now (7) can be represented as

$$\mathbf{\Psi}^+(s) = \mathbf{G}_2^+(s) + \bar{\lambda}\mathbf{R}_\lambda^-(s)\mathbf{G}_2^+(s) + \sum_{k=-N_1}^{N_2} \frac{c_k}{s - z_k} + \chi(s), \quad s \in \mathbb{R}. \tag{14}$$

Applying the inverse Fourier transform to (14) for $t \in \mathbb{R}_+$, we obtain a general solution to (1*) in the form

$$\psi(t) = g(t) + \bar{\lambda} \int_t^\infty \mathbf{r}_{\lambda-}(t-\tau) g(\tau)\, d\tau + \sum_{k=-N_1}^{N_2} c_k \exp(-iz_k t), \quad t \in \mathbb{R}_+. \tag{15}$$

Here the function $\mathbf{r}_{\lambda-}(\theta)$ is the restriction of the Fourier preimage of $\mathbf{R}_\lambda^-(s)$ to the negative semiaxis defined by the relation (see [10])

$$\mathbf{r}_{\lambda-}(\theta) = 2 \sum_{k=-\infty}^{-(N_1+1)} \sqrt{iz_k} \exp(-iz_k\theta) + 2 \sum_{k=N_2+1}^{\infty} \sqrt{iz_k} \exp(-iz_k\theta)$$

$$+ \frac{1}{2\sqrt{\pi}(-\theta)^{3/2}} \sum_{m=1}^{\infty} \frac{m}{\bar{\lambda}^m} \exp\left(\frac{m^2}{4\theta}\right), \quad \operatorname{Re}(iz_k) < 0, \quad |\lambda| > 1, \quad \theta \in \mathbb{R}_-, \tag{16}$$

$$\mathbf{r}_{\lambda-}(\theta) = \frac{1}{2\sqrt{\pi}(-\theta)^{3/2}} \sum_{m=1}^{\infty} m\bar{\lambda}^m \exp\left(\frac{m^2}{4\theta}\right), \quad |\lambda| \le 1, \quad \theta \in \mathbb{R}_-, \tag{17}$$

where the numbers N_1, N_2, and z_k are given by (12) and (8).

Demonstrate (16) and (17). Consider the domain bounded by the integration contour, i.e. closed from above by the semicircle (see Fig. 1). Using the residue theorem and the Jordan lemma (see [10]), we infer

$$\lim_{R\to\infty} \int_{C_R^{(l)}} \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar{\lambda}\exp[-\sqrt{iz}]}\, dz = 0, \quad l = 1, 2,$$

$$\mathbf{r}_{\lambda-}(\theta) = \frac{\lambda \cdot 2\pi i}{2\pi} \sum_{k=-\infty}^{-(N_1+1)} \operatorname{res}\left[z_k, \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar{\lambda}\exp[-\sqrt{iz}]}\right]$$

$$+ \frac{\lambda \cdot 2\pi i}{2\pi} \sum_{k=N_2+1}^{\infty} \operatorname{res}\left[z_k, \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar{\lambda}\exp[-\sqrt{iz}]}\right]$$

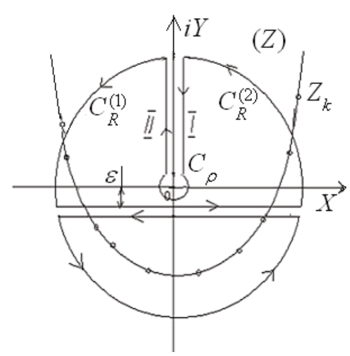

Fig. 1. Integration contours. The points z_k lie on (9).

$$+ \frac{\lambda}{2\pi} \int_{i\infty}^{i\rho} \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar{\lambda}\exp[-\sqrt{iz}]}\, dz + \frac{\lambda}{2\pi} \int_{i\rho}^{i\infty} \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar{\lambda}\exp[-\sqrt{iz}]}\, dz + \frac{\lambda}{2\pi} \int_{C_\rho} \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar{\lambda}\exp[-\sqrt{iz}]}\, dz$$

$$= \frac{1}{2} \sum_{k=-\infty}^{-(N_1+1)} \sqrt{iz_k} e^{-iz_k\theta} + \frac{1}{2} \sum_{k=N_2+1}^{\infty} \sqrt{iz_k} e^{-iz_k\theta} + J_1 + J_2 + J_\rho \quad (\operatorname{Re} iz_k < 0).$$

The integral J_ρ over the small circle C_ρ satisfies the condition $J_\rho \to 0$ as $\rho \to 0$. Indeed, it follows from the relation

$$J_\rho = \frac{\lambda}{2\pi} \int_{c_\rho} \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar{\lambda}\exp[-\sqrt{iz}]}\, dz = \frac{i\lambda\rho}{2\pi} \int_{\frac{5}{2}\pi}^{\frac{\pi}{2}} \frac{\exp[-\sqrt{i\rho e^{i\varphi}} - i\rho e^{i\varphi}\theta]}{1 - \bar{\lambda}\exp[-\sqrt{i\rho e^{i\varphi}}]} e^{i\varphi}\, d\varphi$$

for $|\lambda| > 1$ and the relation

$$J_\rho \approx \frac{i\lambda\rho}{2\pi} \int\limits_{\frac{5}{2}\pi}^{\frac{\pi}{2}} \frac{\exp[-\sqrt{i\rho e^{i\varphi}} - i\rho e^{i\varphi}\theta]}{\sqrt{i\rho e^{i\varphi}}} e^{i\varphi}\, d\varphi$$

for sufficiently small ρ and $|\lambda| = 1$. The sum of the integrals J_1 and J_2 for $|\lambda| > 1$ is written as

$$\begin{aligned} J_1 + J_2 &= \frac{\lambda}{2\pi} \int\limits_{i\infty}^{0} \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar\lambda \exp[-\sqrt{iz}]}\, dz + \frac{\lambda}{2\pi} \int\limits_{0}^{i\infty} \frac{\exp[-\sqrt{iz} - iz\theta]}{1 - \bar\lambda \exp[-\sqrt{iz}]}\, dz \\ &= \frac{i\lambda}{2\pi} \int\limits_{\infty}^{0} \frac{\exp[i\sqrt{u} + u\theta]}{1 - \bar\lambda \exp[i\sqrt{u}]} du + \frac{i\lambda}{2\pi} \int\limits_{0}^{\infty} \frac{\exp[-i\sqrt{u} + u\theta]}{1 - \bar\lambda \exp[-i\sqrt{u}]}\, du \\ &= \frac{i\lambda}{2\pi} \int\limits_{0}^{\infty} \left(\frac{\exp[-i\sqrt{u} + u\theta]}{1 - \bar\lambda \exp[-i\sqrt{u}]} - \frac{\exp[i\sqrt{u} + u\theta]}{1 - \bar\lambda \exp[i\sqrt{u}]} \right) du. \end{aligned}$$

Using the transformation

$$\begin{aligned} \frac{\exp\{-i\sqrt{u}\}}{1 - \bar\lambda \exp\{-i\sqrt{u}\}} - \frac{\exp\{i\sqrt{u}\}}{1 - \bar\lambda \exp\{i\sqrt{u}\}} &= -\frac{1}{\bar\lambda\big(1 - \frac{1}{\bar\lambda}\exp\{i\sqrt{u}\}\big)} + \frac{1}{\bar\lambda\big(1 - \frac{1}{\bar\lambda}\exp\{-i\sqrt{u}\}\big)} \\ = -\frac{1}{\bar\lambda} \sum_{m=1}^{\infty} \frac{1}{\bar\lambda^m} (\exp\{im\sqrt{u}\} - \exp\{-im\sqrt{u}\}) &= -\frac{2i}{\bar\lambda} \sum_{m=1}^{\infty} \frac{1}{\bar\lambda^m} \sin(m\sqrt{u}) \end{aligned}$$

of the integrand, we obtain

$$J_1 + J_2 = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{\bar\lambda^m} \int\limits_0^{\infty} \sin(m\sqrt{u}) \exp\{u\theta\}\, du$$

$$= \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{\bar\lambda^m} \int\limits_0^{\infty} x \sin(mx) \exp\{x^2\theta\}\, dx = \frac{1}{2\sqrt{\pi}(-\theta)^{3/2}} \sum_{m=1}^{\infty} \frac{m}{\bar\lambda^m} \exp\{m^2/(4\theta)\}.$$

In order a solution $\psi(t)$ defined by (15) to be an essentially bounded function, it suffices to require that the integral $\int_t^\infty \mathbf{r}_{\lambda-}(t-\tau)\, d\tau$ be essentially bounded for all $0 < t \le \tau < \infty$, since $g(t) + \sum_{k=-N_1}^{N_2} c_k \exp(-iz_k t)$ is a bounded function of t. This integral is bounded since the function $\mathbf{r}_{\lambda-}(\theta)$ given by (16) satisfies the estimate

$$|\mathbf{r}_{\lambda-}(\theta)| \le C_1|\theta|^{-1/2}\exp(-\delta_0|\theta|) + C_2|\theta|^{-3/2}\exp(-\delta_0|\theta|^{-1}) \quad \forall\theta \in \mathbb{R}_-, \tag{18}$$

where

$$\delta_0 = \min\{1/4;\ [2\pi(N_1+1) + \arg\lambda]^2 - \log^2|\lambda|;\ [2\pi(N_2+1) + \arg\lambda]^2 - \log^2|\lambda|\}. \tag{19}$$

The estimate (18) ensues from the following relations. For the second summand in (16) we have

$$\left| \sum_{k=N_2+1}^{\infty} \sqrt{iz_k} \exp(-iz_k\theta) \right| \le |\log\lambda| \sum_{k=N_2+1}^{\infty} |\exp(-iz_k\theta)|$$

$$\le |\log\lambda| \sum_{k=N_2+1}^{\infty} \exp\{[(2k\pi + \arg\lambda)^2 - \log^2|\lambda|]\theta\} \le \left| \begin{matrix} y = 2k\pi + \arg\lambda \\ a = 2\pi(N_2+1) + \log|\lambda| \end{matrix} \right|$$

$$\leq |\log\lambda| \int_a^\infty \exp\{(y^2 - \log^2|\lambda|)\theta\}\, dy = |\log\lambda| \exp\{-\theta\log^2|\lambda|\} \int_a^\infty \exp\{\theta y^2\}\, dy$$

$$= |z = y - a| = |\log\lambda| \exp\{-\theta\log^2|\lambda|\} \int_0^\infty \exp\{\theta(a^2 + z^2 + 2az)\}\, dz$$

$$= |\log\lambda| \exp\{-\theta\log^2|\lambda| + \theta a^2\} \int_0^\infty \exp\{\theta z^2 + \theta\cdot 2az\}\, dz$$

$$\leq |\log\lambda|(-\theta)^{-1/2} \exp\{\theta(a^2 - \log^2|\lambda|)\} \int_0^\infty \exp\{-(\sqrt{-\theta}z)^2\}\, d(\sqrt{-\theta}z)$$

$$= |\log\lambda| \frac{\sqrt{\pi}}{2\sqrt{-\theta}} \exp\{\delta_2\theta\},$$

where $\delta_2 = [2\pi(N_2+1) + \arg\lambda]^2 - \log^2|\lambda| > 0$.

Similarly, for the first summand we obtain

$$\Bigg|\sum_{k=-\infty}^{-(N_1+1)} \sqrt{iz_k}\exp(-iz_k\theta)\Bigg| \leq |\log\lambda| \frac{\sqrt{\pi}}{2\sqrt{-\theta}} \exp\{\delta_1\theta\},$$

where $\delta_1 = [2\pi(N_1+1) + \arg\lambda]^2 - \log^2|\lambda| > 0$.

The third summand in (16) is estimated as

$$|\theta|^{-3/2} \sum_{m=1}^\infty \frac{m}{\bar\lambda^m} \exp\left(-\frac{m^2}{4|\theta|}\right) = |\theta|^{-3/2} \exp\left(-\frac{1}{4|\theta|}\right) \sum_{m=1}^\infty \frac{m}{\bar\lambda^m} \exp\left(-\frac{m^2-1}{4|\theta|}\right) \leq C|\theta|^{-3/2} \exp\left(-\frac{1}{4|\theta|}\right).$$

The quantity (17) satisfies the estimate

$$|\theta|^{-3/2} \sum_{m=1}^\infty m \exp\left(-\frac{m^2}{4|\theta|}\right) \leq \frac{2}{\sqrt{|\theta|}} \int_1^\infty \exp\left(-\frac{y^2}{4|\theta|}\right) d\left(-\frac{y^2}{4|\theta|}\right) = \frac{2}{\sqrt{|\theta|}} \exp\left(-\frac{1}{4|\theta|}\right)$$

for $|\lambda| = 1$ and the estimate

$$|\theta|^{-3/2} \sum_{m=1}^\infty m\bar\lambda^m \exp\left(-\frac{m^2}{4|\theta|}\right) \leq C|\theta|^{-3/2} \exp\left(-\frac{1}{4|\theta|}\right)$$

for $|\lambda| < 1$.

It is immediate that the function (15) is a solution to (1*) for arbitrary coefficients c_k. Since the number of linearly independent solutions to the corresponding homogeneous equation (1*) is equal to the index $\varkappa^*(\lambda)$ given by (13), (15) is a general solution to the inhomogeneous equation (1*).

First we prove that the function

$$\psi_{hom}(t) = \sum_{k=-N_1}^{N_2} c_k \exp(-iz_k t), \quad t \in \mathbb{R}_+,$$

is a solution to the corresponding homogeneous equation (1*) for every coefficient c_k, $k = 1, 2, \ldots, m$.

Indeed,

$$\sum_{k=-N_1}^{N_2} c_k \exp(-iz_k t) = \sum_{k=-N_1}^{N_2} c_k \frac{\bar{\lambda}}{2\sqrt{\pi}} \int_t^{\infty} \frac{1}{(\tau-t)^{3/2}} \exp\left(-\frac{1}{4(\tau-t)} - iz_k\tau\right) d\tau$$

$$= \sum_{k=-N_1}^{N_2} c_k \frac{\bar{\lambda}}{\sqrt{\pi}} \int_0^{\infty} \exp\left(-\frac{\tau_1^2}{4} - iz_k\left(t + \frac{1}{\tau_1^2}\right)\right) d\tau_1$$

$$= \sum_{k=-N_1}^{N_2} c_k \exp(-iz_k t) \frac{\bar{\lambda}}{\sqrt{\pi}} \int_0^{\infty} \exp\left(-\frac{\tau_1^2}{4} - \frac{iz_k}{\tau_1^2}\right) d\tau_1$$

$$= \sum_{k=-N_1}^{N_2} c_k \exp(-iz_k t)\bar{\lambda} \cdot \exp(-\sqrt{iz_k}) = \sum_{k=-N_1}^{N_2} c_k \exp(-iz_k t).$$

Demonstrate that the function

$$\psi_{part}(t) = g(t) + \bar{\lambda} \int_t^{\infty} \mathbf{r}_{\lambda-}(t-\tau) g(\tau)\, d\tau, \quad t \in \mathbb{R}_+,$$

defines a particular solution to the inhomogeneous equation (1*). Indeed, inserting this function into (1*), we infer

$$g(t) + \int_t^{\infty} \mathbf{r}_{\lambda-}(t-\tau) g(\tau)\, d\tau - \bar{\lambda} \int_t^{\infty} \frac{1}{2\sqrt{\pi}(\tau-t)^{3/2}} \exp\left\{-\frac{1}{4(\tau-t)}\right\} g(\tau)\, d\tau$$

$$-\bar{\lambda} \int_t^{\infty} \frac{1}{2\sqrt{\pi}(\tau-t)^{3/2}} \exp\left\{-\frac{1}{4(\tau-t)}\right\} \left[\int_{\tau}^{\infty} \mathbf{r}_{\lambda-}(\tau-s) g(s) ds\right] d\tau = g(t).$$

Hence, the function $\psi_{part}(t)$ is a particular solution to (1*) if and only if the function $\mathbf{r}_{\lambda-}(-t)$ satisfies the equality

$$\mathbf{r}_{\lambda-}(-t) = \bar{\lambda} \frac{1}{2\sqrt{\pi} t^{3/2}} \exp\{-1/(4t)\} + \bar{\lambda} \int_0^{\infty} \frac{1}{2\sqrt{\pi}\tau^{3/2}} \exp\{-1/(4\tau)\} \mathbf{r}_{\lambda-}(\tau - t)\, d\tau.$$

Calculating the integral in this equality, we have

$$J(t) = \bar{\lambda} \int_0^{\infty} \frac{1}{2\sqrt{\pi}\tau^{3/2}} \exp\{-1/(4\tau)\} \mathbf{r}_{\lambda-}(\tau-t)\, d\tau,$$

where the function $\mathbf{r}_{\lambda-}(\tau - t)$ can be transformed to the form (see Fig. 1)

$$\mathbf{r}_{\lambda-}(\tau-t) = \frac{\bar{\lambda}}{2\pi} \int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} \frac{\exp\{-\sqrt{iz}\}}{1-\bar{\lambda}\exp\{-\sqrt{iz}\}} \exp\{-iz(\tau-t)\}\, dz, \quad \tau < t.$$

For $J(t)$, we obtain

$$J(t) = \frac{\bar{\lambda}^2}{4\pi} \int_0^{\infty} \frac{1}{\sqrt{\pi}\tau^{3/2}} \exp\{-1/(4\tau)\} \Big[\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} \frac{\exp\{-\sqrt{iz}\}}{1-\bar{\lambda}\exp\{-\sqrt{iz}\}} \exp\{-iz(\tau-t)\}\, dz\Big]\, d\tau$$

$$= \frac{\bar{\lambda}^2}{2\pi} \int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} \frac{\exp\{-\sqrt{iz}+izt\}}{1-\bar{\lambda}\exp\{-\sqrt{iz}\}} \Big[\frac{1}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{\tau^{3/2}} \exp\{-iz\tau - 1/(4\tau)\}\, d\tau\Big]\, dz,$$

since $\operatorname{Im} z = -\varepsilon$, $\varepsilon > 0$ (see Fig. 1). The inner integral is equal to

$$\frac{1}{2\sqrt{\pi}} \int\limits_0^\infty \frac{1}{\tau^{3/2}} \exp\{-iz\tau - 1/(4\tau)\}\, d\tau = \frac{1}{\sqrt{\pi}} \int\limits_0^\infty \exp\left\{-x^2 - iz/(4x^2)\right\} dx = \exp\{-\sqrt{iz}\}$$

and so

$$J(t) = \frac{\bar{\lambda}^2}{2\pi} \int\limits_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} \frac{\exp\{-2\sqrt{iz} + izt\}}{1 - \bar{\lambda}\exp\{-\sqrt{iz}\}}\, dz.$$

The integrand extends analytically to the domain $\operatorname{Im} z > -\varepsilon$ of the complex plane with the cut along the positive imaginary semiaxis (see Fig. 1).

The residue theorem and the Jordan lemma (see [10]) yield

$$J(t) = 2\bar{\lambda} \sum_{k=-\infty}^{-(N_1+1)} \sqrt{iz_k} \exp\{-\sqrt{iz_k} + iz_k t\} + 2\bar{\lambda} \sum_{k=N_2+1}^{\infty} \sqrt{iz_k} \exp\{-\sqrt{iz_k} + iz_k t\}$$

$$+ \frac{\bar{\lambda}^2 i}{2\pi} \int\limits_0^\infty \left[\frac{\exp\{-2i\sqrt{u} - ut\}}{1 - \bar{\lambda}\exp\{-i\sqrt{u}\}} - \frac{\exp\{2i\sqrt{u} - ut\}}{1 - \bar{\lambda}\exp\{i\sqrt{u}\}} \right] du.$$

Transforming the integrand in the last integral to the form

$$\frac{\exp\{-2i\sqrt{u}\}}{1 - \bar{\lambda}\exp\{-i\sqrt{u}\}} - \frac{\exp\{2i\sqrt{u}\}}{1 - \bar{\lambda}\exp\{i\sqrt{u}\}} = -\frac{\exp\{-i\sqrt{u}\}}{\bar{\lambda}\left(1 - \frac{1}{\bar{\lambda}}\exp\{i\sqrt{u}\}\right)} + \frac{\exp\{i\sqrt{u}\}}{\bar{\lambda}\left(1 - \frac{1}{\bar{\lambda}}\exp\{-i\sqrt{u}\}\right)}$$

$$= -\frac{1}{\bar{\lambda}} \sum_{m=0}^\infty \left[\frac{\exp\{-i\sqrt{u}\}}{\bar{\lambda}^m} \exp\{im\sqrt{u}\} - \frac{\exp\{i\sqrt{u}\}}{\bar{\lambda}^m} \exp\{-im\sqrt{u}\} \right]$$

$$= \frac{2i}{\bar{\lambda}} \sin\sqrt{u} - \frac{2i}{\bar{\lambda}^2} \sum_{m=1}^\infty \frac{1}{\bar{\lambda}^m} \sin(m\sqrt{u}),$$

we obtain

$$J(t) = 2 \sum_{k=-\infty}^{-(N_1+1)} \sqrt{iz_k} \exp\{iz_k t\} + 2 \sum_{k=N_2+1}^{\infty} \sqrt{iz_k} \exp\{iz_k t\}$$

$$+ \int\limits_0^\infty \left[-\frac{\bar{\lambda}}{\pi} \exp\{-ut\} \sin\sqrt{u} + \frac{1}{\pi} \sum_{m=1}^\infty \frac{\sin(m\sqrt{u})}{\bar{\lambda}^m} \exp\{-ut\} \right] du$$

$$= \mathbf{r}_{\lambda-}(-y) - \frac{\bar{\lambda}}{2\sqrt{\pi} y^{3/2}} \exp\{-1/(4t)\}.$$

Here we use the equalities $\exp(-\sqrt{iz_k}) = 1/\bar{\lambda}$ and $z_k = -i[\log^2|\lambda| - (2k\pi - \arg\lambda)^2] - 2\log^2|\lambda|(2k\pi - \arg\lambda)$ and the series expansion of the integrand in the powers of $1/\bar{\lambda}$.

Thus, we have established the required equality and the function $\psi_{part}(t)$ defines a particular solution to the inhomogeneous equation (1*). The function (15) is a general solution to (1*) for all coefficients c_k, $k = 1, 2, \ldots, m$. Thereby we established the following lemmas:

Lemma 1. *The values $\lambda \in D_0$ in* (11) *are regular numbers of k^*_λ* (*see* (1*)).

Lemma 2. *The set $\mathbb{C} \setminus D_0$ consists of the characteristic numbers of k^*_λ* (*see* (1*)). *Moreover, if*

$$\lambda \in D_m \cup \Gamma_{m-1} \setminus \{(-1)^m e^{m\pi}\}, \quad m = 1, 2, \ldots,$$

then $\dim\operatorname{Ker}(k^*_\lambda) = \varkappa^*(\lambda) = m$ *and the corresponding eigenfunctions are of the form*

$$\psi_{\lambda k}(t) = \exp(-iz_k t), \quad k = 1, \ldots, m = \varkappa^*(\lambda) = N_1 + N_2 + 1.$$

3. Solving the integral equation (1). Examine (4). Applying the Fourier transform, we infer

$$\Phi^+(s) - \lambda \mathbf{K}^+(s)\Phi^+(s) = \mathbf{F}_2^+(s) + \Phi^-(s), \quad s \in \mathbb{R}, \tag{20}$$

where the capitals stand for Fourier images.

If

$$\mathbf{A}_\lambda(s) \equiv 1 - \lambda \mathbf{K}^+(s) \neq 0 \quad \forall s \in \mathbb{R}, \tag{21}$$

then in view of (20) we arrive at the Riemann problem [7–9]

$$\Phi^+(s) = [\mathbf{A}_\lambda(s)]^{-1}\Phi^-(s) + \lambda \mathbf{R}_\lambda^+(s)\mathbf{F}_2^+(s) + \mathbf{F}_2^+(s), \quad s \in \mathbb{R}, \tag{22}$$

where $\mathbf{R}_\lambda^+(s) = \mathbf{K}^+(s)/\mathbf{A}_\lambda(s)$. The coefficient $\mathbf{R}_\lambda^+(s)$ in (22) can be analytically extended to the upper half-plane with the possible exception of finitely many poles which are the zeros of the function $\mathbf{A}_\lambda(s)$. Moreover, the index $\varkappa(\lambda)$ of (22) is nonpositive, i.e., $\varkappa(\lambda) = -\varkappa^*(\lambda) \leq 0$. Rewriting (20) as

$$[1 - \lambda \mathbf{K}^+(s)]\Phi^+(s) = \mathbf{F}_2^+(s) + \Phi^-(s), \quad s \in \mathbb{R},$$

we obtain $\Phi^-(s) \equiv 0$; thus, (22) takes the form

$$\Phi^+(s) = \lambda \mathbf{R}_\lambda^+(s)\mathbf{F}_2^+(s) + \mathbf{F}_2^+(s), \quad s \in \mathbb{R}. \tag{23}$$

The last relation implies that the homogeneous integral equation corresponding to (1) has only the trivial solution for all $\lambda \in \mathbb{C}$.

Applying the inverse Fourier transform to (23) for $t \in \mathbb{R}_+$, we obtain a solution to (1) as follows:

$$\varphi(t) = f(t) + \lambda \int_0^t \mathbf{r}_{\lambda+}(t-\tau) f(\tau)\, d\tau, \quad t \in \mathbb{R}_+, \tag{24}$$

where $\mathbf{r}_{\lambda+}(\theta)$ is the restriction of the preimage of the Fourier transform $\mathbf{R}_\lambda^+(s)$ to the positive semiaxis (see [10]) which is defined by the formula

$$\mathbf{r}_{\lambda+}(\theta) = 2 \sum_{k=-\infty}^{-(N_1+1)} \sqrt{-iz_k} \exp(iz_k\theta) + 2 \sum_{k=N_2+1}^{\infty} \sqrt{-iz_k} \exp(iz_k\theta)$$

$$+ \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} \frac{m}{\lambda^m} \exp\left(-\frac{m^2}{4\theta}\right), \quad \mathrm{Re}(-iz_k) > 0, \ |\lambda| > 1, \quad \theta \in \mathbb{R}_+, \tag{25}$$

$$\mathbf{r}_{\lambda+}(\theta) = \frac{1}{2\sqrt{\pi}\theta^{3/2}} \sum_{m=1}^{\infty} m\lambda^m \exp\left(-\frac{m^2}{4\theta}\right), \quad |\lambda| \leq 1, \quad \theta \in \mathbb{R}_+;$$

here N_1, N_2, and z_k are the numbers from (12) and (8).

In order the function $\varphi(t)$ in (24) to be integrable, it is sufficient that the function $\mathbf{r}_{\lambda+}(t-\tau)$ be bounded for arbitrary $0 < \tau \leq t < \infty$, since the function $f(t)$ is integrable. The function $\mathbf{r}_{\lambda+}(t-\tau)$ is bounded, since the function $\mathbf{r}_{\lambda+}(\theta)$ in (25) admits the estimate

$$|\mathbf{r}_{\lambda+}(\theta)| \leq C_1|\theta|^{-1/2}\exp(-\delta_0|\theta|) + C_2|\theta|^{-3/2}\exp(-\delta_0|\theta|^{-1}) \quad \forall \theta \in \mathbb{R}_+,$$

with the constant δ_0 defined by (19).

Next, if $\lambda \in D_0$ then by Lemma 1 (1) is unconditionally and uniquely solvable; if $\lambda \in \mathbb{C} \setminus D_0$ then by Lemma 2 for the unique solvability of (1), it is necessary and sufficient to have the orthogonality conditions

$$\int_0^\infty \overline{f(t)} \exp(-iz_k t)\, dt = 0, \quad k = 1, \ldots, m.$$

We have proven the following lemma.

Lemma 3. *There are no characteristic numbers of* k_λ (1) *on* $\mathbb{C}$.

Lemmas 1–3 imply the following main result of this article stated in the form of a theorem.

Theorem. *The problem* (1) *is Noetherian and the index of* k_λ *is as follows:*

$$\mathrm{Ind}(k_\lambda) = \dim \mathrm{Ker}(k_\lambda) - \dim \mathrm{Ker}(k_\lambda^*) = -\varkappa^*(\lambda) = -N_1 - N_2 - 1.$$

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